AD-A069 146 DAVID W TAYLOR NAVAL SHIP RESEARCH AND DEVELOPMENT CE--ETC F/G 17/1
THEORETICAL AND EXPERIMENTAL EVALUATION OF POLYA PROCESSES.(U)
APR 70 P SINEX, R WASSERMAN

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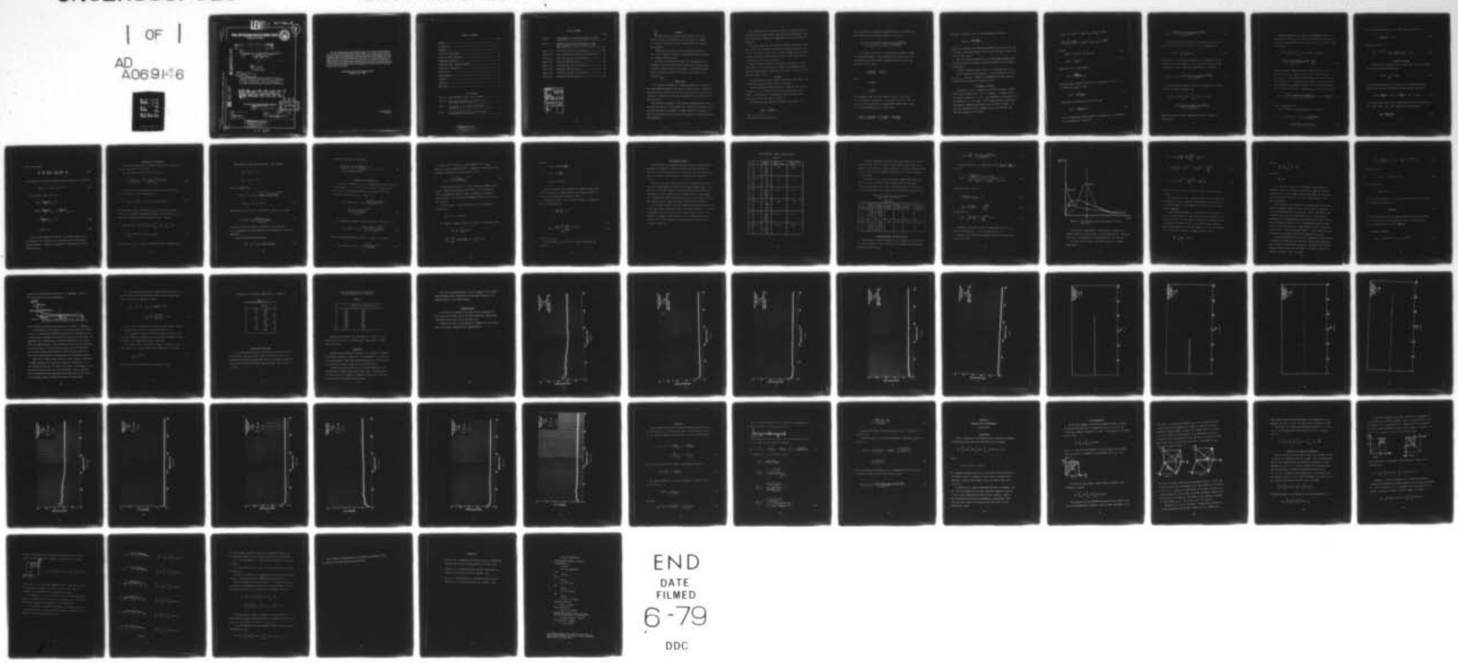

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NAVAL SHIP RESEARCH AND DEVELOPMENT CENTER

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DAV-294-931

AD A069146

(6) THEORETICAL AND EXPERIMENTAL EVALUATION OF PÓLYA PROCESSES,

by

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(12) 60 p.

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387 694

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DEPARTMENT OF ACOUSTICS AND VIBRATION
(9) TECHNICAL NOTE,

(11) APRIL 1970

(14) DTNSRDC/DAV-294-931

THEORETICAL AND EXPERIMENTAL EVALUATION OF PÓLYA PROCESSES
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387 694

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Washington, D.C. 20007

NDW NSRDC 3960/46
(REV 6-69)

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ABSTRACT

The Pólya process, in which the occurrence of one event affects the time of occurrence of the next event, was studied as a possible description of the clustering found in submarine echo structure.

Theoretical derivations of the properties of the Pólya process are presented, followed by experimental results obtained by a computer simulated model.

The Pólya process has been found to be approximated by an ensemble of Poisson processes with random mean, with pronounced clustering appearing only in the ensemble. This representation makes it difficult to employ the process in the study of echo structures.

INTRODUCTION

It was initially thought that a clustering formation in echo structure from submarines might be described by a Pólya process, in which the occurrence of one event affects the time of occurrence of the next event, (leading to a bunching of events more pronounced than clustering due to a Poisson process).

The experiments performed, using computer simulated Pólya processes, show that the process is a non-ergodic one. Parameter estimation is thus very difficult unless an ensemble of sample functions is considered. A model for the Pólya process was found in an ensemble of Poisson processes with random mean.

Most significant was the discovery that the clustering effect of the process is an ensemble rather than sample function property. Thus the process would be difficult to employ practically in this physical situation.

The model consisting of an ensemble of Poisson processes is applicable as the time and number of events become large (usually $N > 100$ events). However, the first segment of any one sample function, which is marked by pronounced fluctuations in mean, could contain useful information and is still under study.

Theoretical derivations of the various characteristics of the Pólya process are first discussed, followed by experimental results obtained through computer simulation. This paper is in part an elaboration of Conolly's[1] work.

URN MODEL

The Pólya process is most clearly introduced by considering first the discrete situation represented by an urn model.

From an urn having b black and w white balls, one ball is drawn at random. It is returned to the urn with s balls of the color chosen. The probability of a particular color being drawn thus increases each time that color is chosen.

The probability of drawing two successive black balls is

$$\frac{b}{b + w} \quad \frac{b + s}{b + w + s}$$

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[1] References are listed on page 55.

and, in general, the probability of choosing, out of n drawings, n_1 black balls, and n_2 white balls (where $n = n_1 + n_2$) is

$$\frac{b(b + s)\ldots(b + \overline{n_1 - 1}\, s)\; w(w + s)\ldots(w + \overline{n_2 - 1}\, s)}{(b + w)\;(b + w + s)\ldots(b + w + \overline{n - 1}\, s)}$$

If one defines the state of the system as the number of black balls drawn and denotes a system having k black balls by E_k, then the probability of there being a transition from state E_k to E_{k+1} at the $(n + 1)^{th}$ trial is

$$\frac{b + ks}{b + w + ns} = \frac{p + k\gamma}{1 + n\gamma}$$

where

$$p = \frac{b}{b + w}$$

$$\gamma = \frac{s}{b + w}$$

Let drawings be made such that each one occurs in time h where $h = t/n$ and let $h \to 0$ while $n \to \infty$ so that $np \to \lambda t$ and $n\gamma \to a\lambda t$. The factor a can be considered a "contagiousness" factor and is equal to 0 for the Poisson process. In this limit

$$\frac{p + k\gamma}{1 + n\gamma} = \frac{1}{n}\,\frac{np + kn\gamma}{1 + n\gamma} = \frac{1}{n}\,\frac{\lambda t + ka\lambda t}{1 + a\lambda t} = \frac{\lambda(1 + ka)}{1 + a\lambda t}\, h$$

Letting k = n where n = 0, 1, 2...and replacing h by Δt gives

$$\beta_n(t)\ \Delta t = \frac{\lambda(1 + an)}{1 + a\lambda t}\ \Delta t$$

and $\beta_n(t)\ \Delta t$ expresses the conditional probability of an event (in lieu of drawing a black ball) during the time interval Δt after n events have occurred during time t.[2]

Thus the transition probability in the Pólya process depends upon the actual state of the system, a situation described in the following two postulates:

1. Direct transition from a state E_n is only permitted to E_{n+1};
2. The probability that during (t,t + h) a transition occurs from E_n to E_{n+1} is $[\beta_n\ \Delta t + o(h)]$ and the probability that more than one transition occurs during this interval is o(h).

CONTINUOUS SITUATION

In time (0, t + h), n changes can take place by having n changes occur in (0,t) and none in (t, t + h), or having n - 1 changes occur in (0,t) and one change in (t, t + h). The probability of having more than one change in (t, t + h) is o(h). Letting $P_n(t)$ denote the probability that exactly n changes occur during a time interval of length t, and $P_o(t)$ the probability that no changes occur in this interval, the following equations can be formed:

$$P_n(t + h) = P_n(t)\ (1 - \beta_n h) + P_{n-1}(t)\ \beta_{n-1} h + o(h)$$

$$\frac{P_n(t + h) - P_n(t)}{h} = - \beta_n P_n(t) + \beta_{n-1}\ P_{n-1}(t) + \frac{o(h)}{h}$$

therefore

$$P_n'(t) = - \beta_n P_n(t) + \beta_{n-1}\ P_{n-1}(t) \tag{1}$$

$$P_o'(t) = - \beta_o P_o(t) \tag{2}$$

Equation (2) can be solved giving

$$P_o(t) = \frac{\text{CONST}}{(1 + a\lambda t)^{1/a}}$$

Using the initial condition that no event takes place at time $t = 0$, it follows that $P_o(0) = 1$ and

$$P_o(t) = \frac{1}{(1 + a\lambda t)^{1/a}} \tag{3}$$

The solution of Equation (1) for $P_1(t)$ is then

$$P_1(t) = \frac{\lambda t}{(1 + a\lambda t)^{1+1/a}} \tag{4}$$

with the integration constant chosen so that $P_1(0) = 0$. By induction it can be shown that in general[3]

$$P_n(t) = \frac{\lambda^n t^n (1 + a)\,(1 + 2a)\ldots(1 + \overline{n - 1}\,a)}{n!\,(1 + a\lambda t)^{n+1/a}} \tag{5}$$

The probability density $P_1(t_1)$ of having only one event in the interval $(0,t)$ with time of occurrence t_1 is (see Appendix A)

$$p_1(t_1) = \frac{\lambda}{(1 + a\lambda t)^{1+1/a}} \tag{6}$$

This density is uniform and independent of t_1; it depends only upon the total time being considered. Thus this event can occur at any random time t_1 within t with uniform probability. In general,

$$p(t_1, t_2, \ldots t_n) = \frac{\lambda^n (1 + a)\,(1 + 2a)\ldots(1 + \overline{n - 1}\,a)}{(1 + a\lambda t)^{n+1/a}} \tag{7}$$

The probability of having n events occur anywhere in the interval $(0,t)$ is then (see Appendix B)

$$P_n(t) = \int_0^t dt_1 \int_{t_1}^t dt_2 \ldots \int_{t_{n-1}}^t p(t_1, t_2 \ldots t_n)\, dt_n$$

$$= \frac{\lambda^n t^n\,(1 + a)\,(1 + 2a)\ldots(1 + \overline{n - 1}\,a)}{n!\,(1 + a\lambda t)^{n+1/a}}$$

which has been found earlier [Equation (5)] as the solution to Equation (1).

Consider now $P_{n+1}(t_1,t_2,\ldots t_{n+1})$, the probability that $n + 1$ events occur in time t_{n+1}, the last event occurring at the end of the interval. Integrating with respect to $t_1,t_2,\ldots t_{n-1}$ over their ranges will result in

$$\int_0^{t_n} dt_1 \int_{t_1}^{t_n} dt_2 \ldots \int_{t_{n-2}}^{t_n} P_{n+1}\ (t_1,t_2,\ldots t_{n+1})\, dt_{n-1}$$

$$= \frac{\lambda^{n+1}\ (1 + a)(1 + 2a)\ldots(1 + na)}{(1 + a\lambda t_{n+1})^{n+1+1/a}}\ \frac{t_n^{\,n-1}}{(n-1)!} \tag{8}$$

which is the joint probability density that the n^{th} event will occur in the interval $(t_n, t_n + dt_n)$ and the $(n + 1)^{th}$ in the interval $(t_n + \tau, t_n + \tau + d\tau)$, despite the particular times of occurrence of the previous events. Integrating with respect to t_n over its range $(0,\infty)$ will give the probability density of τ, the interval between the n^{th} and $(n + 1)^{th}$ events. Replacing t_{n+1} by $t_n + \tau$ and integrating

$$g(\tau) = \int_0^{\infty} \frac{\lambda^{n+1}\ (1 + a)\ldots(1 + na)\ t_n^{\,n-1}}{(n - 1)!\ (1 + a\lambda t_n + a\lambda\tau)^{n+1+1/a}}\, dt_n = \frac{\lambda}{(1 + a\lambda\tau)^{1+1/a}} \tag{9}$$

which is independent of n.

The r^{th} moment of the distribution of τ is given by

$$\mu_r = \int_0^{\infty} \tau^r\, g(\tau)\, d\tau = \int_0^{\infty} \tau^r\, \frac{\lambda}{(1 + a\lambda\tau)^{1+1/a}}\, d\tau$$

$$= \frac{r!}{\lambda^r\ (1 - a)(1 - 2a)\ldots(1 - ra)} \tag{10}$$

with the restriction that $1/a > r$. Thus the mean value of τ is

$$\bar{\tau} = \frac{1}{\lambda(1 - a)} , \quad a < 1 , \tag{11}$$

and the variance is

$$\sigma^2(\tau) = \overline{\tau^2} - \bar{\tau}^2 = \frac{1}{\lambda^2(1 - a)^2 \, (1 - 2a)} , \quad a < \frac{1}{2} \tag{12}$$

GENERATING FUNCTION

If $P_n(t)$ is the probability that n events occur in the time interval (0,t) then the generating function is

$$P(x,t) = \sum_{n \geq 0} x^n P_n(t) \tag{13}$$

By differentiating P(x,t) with respect to t and using differential Equations (1) and (2) which define the Pólya process one obtains

$$(1 + a\lambda t) \frac{\delta P(x,t)}{\delta t} + \lambda a x(1 - x) \frac{\delta P(x,t)}{\delta x} = \lambda(x - 1) P(x,t) \tag{14}$$

The left-hand side represents the derivative of P(x,t) in the direction $((1 + a\lambda t), \lambda a x(1 - x))$. Thus, along the curve described by

$$\frac{dx}{dt} = \frac{\lambda a x(1 - x)}{1 + a\lambda t} , \tag{15}$$

P(x,t) will satisfy

$$\frac{dP}{dx} = \frac{\delta P}{\delta x} + \frac{\delta P}{\delta t}\frac{dt}{dx} = \frac{\lambda(x - 1)\ P}{\lambda a x(1 - x)} = \frac{-P}{ax} \tag{16}$$

Solving Equations (15) and (16) simultaneously with $P(1,t) = 1$ will give

$$P(x,t) = (1 + a\lambda(1 - x)t)^{-1/a} \tag{17}$$

The mean and variance of n are:

$$E(n) = \frac{\delta P(x,t)}{\delta x}\bigg|_{x = 1} = \lambda t$$

$$Var(n) = \frac{\delta^2 P(x,t)}{\delta x^2}\bigg|_{x = 1} + \frac{\delta P(x,t)}{\delta x}\bigg|_{x = 1}$$

$$- \left(\frac{\delta P(x,t)}{\delta x}\bigg|_{x = 1}\right)^2 \tag{18}$$

$$= a\lambda^2 t^2 + \lambda t \tag{19}$$

Thus, the mean number of events, for an ensemble average, in the time interval t is λ, which also is the mean for the Poisson process having intensity λ. However, the variance is larger than that of the Poisson process.

CORRELATION OF INTERVALS

An interesting conclusion is that the correlation coefficient of two time intervals is a, if $a < \frac{1}{2}$.

The correlation coefficient of τ_j and τ_{n+1} is

$$\rho = \frac{\mathrm{cov}(\tau_j, \tau_{n+1})}{\sigma(\tau_j)\ \sigma(\tau_{n+1})} = \frac{E(\tau_j\ \tau_{n+1}) - E(\tau_j)\ E(\tau_{n+1})}{\sigma(\tau_j)\ \sigma(\tau_{n+1})} \tag{20}$$

Each term has previously been found [Equations (11) and (12)] except $E(\tau_j\ \tau_{n+1})$. This is expressed as

$$E(\tau_j\ \tau_{n+1}) = \int_0^\infty \int_0^\infty \tau_j\ \tau_{n+1}\ p(\tau_j, \tau_{n+1})\ d\tau_j\ d\tau_{n+1} \tag{21}$$

The probability density function $p(\tau_j, \tau_{n+1})$ of the two intervals τ_j and τ_{n+1} is found from the probability density function $p(t_j, \tau_j, t_n, \tau_{n+1})$ that these intervals follow the particular times t_j and t_n

$$p(t_j, \tau_j, t_n, \tau_{n+1}) = \int_0^{t_j} dt_1 \int_{t_1}^{t_j} dt_2 \ldots \int_{t_{j-2}}^{t_j} dt_{j-1} \int_{t_{j+1}}^{t_n} dt_{j+2} \ldots$$

$$\int_{t_{n-2}}^{t_n} p(t_1, t_2, \ldots t_n, t_n + \tau_{n+1})\ dt_{n-1} \tag{22}$$

where $p(t_1, t_2, \ldots t_n, t_n + \tau_{n+1})$ is the uniform density of Equation (7).

Substituting in each term greater than t_j the variable

$$t'_{j+2} = t_{j+2} - \tau_j - t_j$$

$$t'_{j+3} = t_{j+3} - \tau_j - t_j$$

gives (see Appendix B)

$$p(t_j, \tau_j, t_n, \tau_{n+1}) = \frac{t_j^{\,j-1}}{(j-1)!} \frac{(t_n - \tau_j - t_j)^{n-j-2}}{(n - j - 2)!}$$

$$p(t_1, t_2, \ldots t_n, t_n + \tau_{n+1}) \tag{23}$$

Integrating t_j from 0 to $(t_n - \tau_j)$ and then t_n from τ_j to ∞ yields

$$p(\tau_j, \tau_{n+1}) = \frac{\lambda^2 (1 + a)}{(1 + a\lambda(\tau_j + \tau_{n+1}))^{2+1/a}} \tag{24}$$

This result is just the density of two events in the interval $\tau_j + \tau_{n+1}$ [Equation (7)] and exemplifies the independence of time (see Appendix A).

Integrating Equation (21) then gives

$$E(\tau_j \ \tau_{n+1}) = \frac{1}{\lambda^2} \left(\frac{1}{1 - a}\right) \left(\frac{1}{1 - 2a}\right) \tag{25}$$

Combining terms gives the correlation

$$\rho = \frac{\frac{1}{\lambda^2}\left(\frac{1}{1-a}\right)\left(\frac{1}{1-2a}\right)-\left[\frac{1}{\lambda}\left(\frac{1}{1-a}\right)\right]^2}{\left[\left(\frac{1}{\lambda^2}\left(\frac{1}{1-2a}\right)\left(\frac{1}{1-a}\right)^2\right)^{\frac{1}{2}}\right]^2} = a, \text{ if } a < \frac{1}{2} \tag{26}$$

GENERATING PÓLYA PROCESSES

To generate a Pólya process it is advantageous to utilize a method given by Conolly.[3] If n events have occurred in the time interval $(0, t_n)$, then the conditional probability for the $(n + 1)^{th}$ event occurring in interval $(t_{n+1}, t_{n+1} + dt_{n+1})$ is

$$h_{n+1}\,(t_{n+1}|t_1,t_2,\ldots t_n) = \frac{p_{n+1}\,(t_1,t_2,\ldots t_{n+1})}{p_n(t_1,t_2,\ldots t_n)}$$

$$= \frac{\lambda(1 + na)\,(1 + a\lambda t_n)^{n+1/a}}{(1 + a\lambda t_{n+1})^{n+1+1/a}} \tag{27}$$

The conditional probability density of the $(n + 1)^{th}$ interval τ is

$$h_{n+1}\,(\tau|t_1,t_2,\ldots t_n) = \frac{\lambda(1 + na)(1 + a\lambda t_n)^{n+1/a}}{(1 + a\lambda t_n + a\lambda\tau)^{n+1+1/a}} \tag{28}$$

which, when integrated between 0 and T, the range of τ, becomes

$$H_{n+1}\,(T|t_1,t_2,\ldots t_n) = 1 - \left(\frac{1 + a\lambda t_n}{1 + a\lambda t_n + a\lambda\tau}\right)^{n+1/a} \tag{29}$$

The $(n+1)^{th}$ interval τ_{n+1} can be computed if $(1 - H_{n+1}(T|t_1,t_2,\ldots t_n))$ is replaced by one of a sequence of random numbers (R) uniformly distributed over the interval $0 < R < 1$.

$$R_{n+1} = \left[\frac{1 + a\lambda t_n}{1 + a\lambda t_n + a\lambda\tau_{n+1}}\right]^{n+1/a} \tag{30}$$

To show that this procedure is valid, consider a random variable x with a uniform distribution between 0 and 1. It is necessary to find a transformation function G which transforms x into another random variable τ with the required probability density $p_\tau(\tau)$ of the next interval.

If the relation $\tau = G(x)$ is defined, then the transformation relations are

$$F(\tau) = x \longleftrightarrow G(x) = \tau$$

The relation between p_x and p_τ is found in the following manner:

$$P(x \le \alpha) = \int_o^\alpha p_x(x)\, dx$$

$$= \int_{\tau_o = G(o)}^{\tau_\alpha = G(\alpha)} p_x(x(\tau)) \left|\frac{dx}{d\tau}\right| d\tau = \int_{\tau_o}^{\tau_\alpha} p_\tau(\tau)\, d\tau$$

therefore

$$p_\tau(\tau) = p_x(x(\tau))\left|\frac{dx}{d\tau}\right|$$

$$p_\tau(\tau) = \frac{1}{1}\ \frac{dF(\tau)}{d\tau}$$

$$\int p_\tau(\tau)\ d\tau = F(\tau) = x$$

Thus, the function that transforms the random variable x with uniform density into a random variable τ with density $p_\tau(\tau)$ is the inverse of the distribution function of that τ.

From Equation (30) a set of recurrent equations is formed with which τ_n can be generated.

$$\tau = \left(\frac{1}{a\lambda}\right)\left(\frac{1}{R^a} - 1\right)$$

.

.

. (31)

$$\tau_{n+1} = \left(\frac{1}{a\lambda}\right)\left(\frac{1 + a\lambda t_n}{R_{n+1}^{\frac{a}{an+1}}} - 1 - a\lambda t_n\right) \qquad (32)$$

with $t_n = t_{n-1} + \tau_n$.

The Pólya process is then the set of these successive time intervals.

EXPERIMENTAL RESULTS

Pólya processes were generated through a computer simulation using Equations (31) and (32). The first 500 events of each process were examined. Several characteristics of the process were studied through the use of graphs, and the results of variations of the parameters a and λ.

Earlier it was shown that the mean value of an interval τ between successive events is $\frac{1}{\lambda(1 - a)}$ and the variance $\frac{1}{\lambda^2 (1 - a)^2 (1 - 2a)}$. The results of an extensive number of trials show clearly that the process is non-ergodic, as the ensemble means do not agree with the sample means. The following table which shows the means across several sample functions and the mean of these sample means will clarify this non-ergodicity.

As is obvious from the data, the theoretical and experimental values tend to vary greatly from each other as the factor a reaches .5, an effect in accordance with the fact that the variance increases with a. The variance is no longer a valid measure if a > .5 and the mean loses its validity if a $\geq$ 1, as theoretically each will go to infinity at these respective values of a.

MEAN INTERVALS: Sample Versus Ensemble

TABLE 1A

a	λ	Sample Means	Mean of Sample Means	Theoretical Mean
.3	.5	1.31 1.89 2.01 7.82 3.59	3.32	2.86
.3	2.0	.53 .76 .62 .31 .33	.51	.71
.4	.5	3.13 2.38 10.78 1.82 1.15	3.85	3.33
.4	2.0	.27 3.08 .37 .49 .26	.89	.83
.5	.5	.87 2.87 2.16 5.04 1.18	2.42	4.00
.5	2.0	.44 .37 8.76 1.17 .79	2.31	1.00

In another experiment the factor a was given values larger than .5 and 1.0 so that either the mean or variance would theoretically go to infinity. The results are shown in Table 1B. The values do become large; however, they will never approach infinity due to the finite bounds of the experiment.

From Figures 3-7 it can be seen that the sample means approach a constant value very quickly, usually within the first 100 events. But since these means are in such a large spread about the theoretical mean, the parameters a and λ cannot be determined from a single process.

Means & Variances of the Interval with
Large Values for Factor a

TABLE 1B

a	λ	Sample Mean	Sample Variance	Ensemble Mean	Ensemble Variance	Theoretical Mean	Theoretical Variance
.5	4/30	14.16 8.41 9.88 9.57 8.23	198.14 68.16 100.47 95.81 70.50	10.05	106.61	15.0	Infinity
1.0	4/30	27.25 6.12 6.66 21.83 47.27	710.07 36.11 38.53 457.42 2239.84	21.82	696.39	Infinity	Infinity

APPROXIMATE MODEL FOR PÓLYA PROCESS

The probability density function for β_∞, the mean rate of intensity as t and n approach ∞, of the Pólya process is found by transforming the probability density $p_n(t)$.

$$p_n(t) = \frac{\lambda^n t^n}{n!} \quad \frac{(1 + a)\ldots(1 + \overline{n - 1}\,a)}{(1 + a\lambda t)^{n+1/a}} \tag{33}$$

Using the substitution $t = \frac{n}{\beta}$ and the Jacobian, $\frac{n}{\beta}(p_\beta(\beta) = \left|\frac{dn}{d\beta}\right| p_n(n)$, results in

$$p_\beta(\beta) = \frac{\lambda^n \left(\frac{n}{\beta}\right)\left(\frac{n}{\beta}\right)^n \; (1 + a)\ldots(1 + na)}{n!\,n^{n+1/a} \; \left(\frac{a\lambda}{\beta}\right)^{n+1/a} \; \left(1 + \frac{\beta}{a\lambda n}\right)^{n+1/a} \; (1 + na)} \tag{34}$$

Upon employing the identity

$$\frac{n^z \; n!}{z(z + 1)\ldots(z + n)} = \Gamma(z) \tag{35}$$

$$p_\beta(\beta) = \frac{1}{a\lambda} \left(\frac{\beta}{a\lambda}\right)^{1/a-1} \quad \frac{e^{-\beta/a\lambda}}{\Gamma\left(\frac{1}{a}\right)} \tag{36}$$

The mean $E(\beta)$ is

$$E(\beta) = \int_o^\infty \frac{\beta}{a\lambda} \left(\frac{\beta}{a\lambda}\right)^{1/a-1} \frac{e^{-\beta/a\lambda}}{\Gamma\left(\frac{1}{a}\right)} \, d\beta = \lambda \tag{37}$$

Although the distribution of β_∞ is spread about the value λ as evidenced by Figures 12b - 14b, the mean value is λ.

A graph of the probability distribution of β for various values of a is given below.

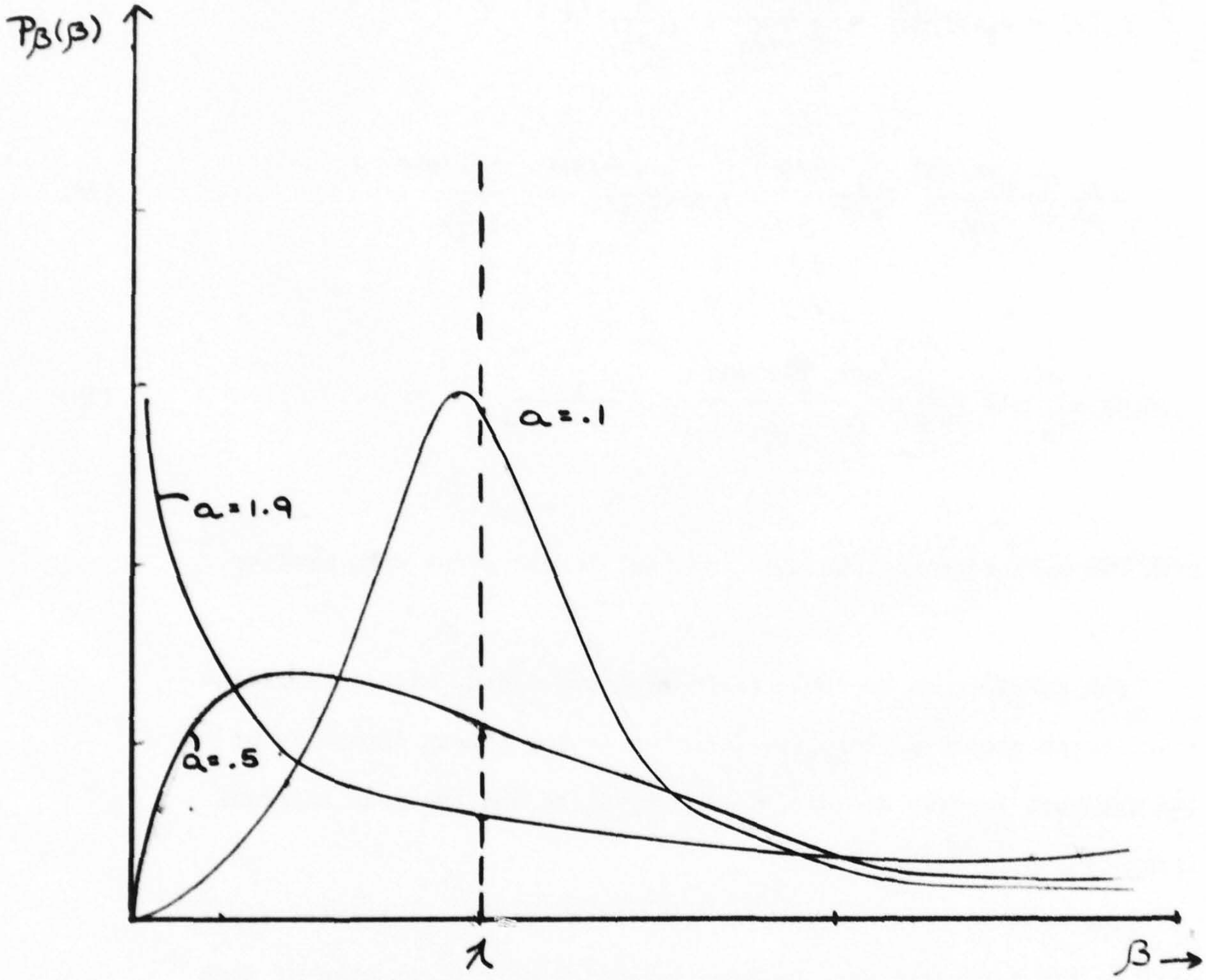


Figure 1 - The probability distribution $p_\beta(\beta)$ for various values of a

As the factor a approaches 0, the distribution approaches the singular value of λ, characteristic of the Poisson process. For larger values of a, the intensities are spread about the mean intensity λ.

The mean interval length $\tau = \frac{t}{n}$ can be found by calculating $p_\tau(\tau)$ using

$$p_\tau(\tau) = p_\beta(\beta)\left|\frac{d\beta}{d\tau}\right| = \frac{\beta^2}{a\lambda}\,\frac{e^{-\beta/a\lambda}}{\Gamma(\frac{1}{a})}\,\left(\frac{\beta}{a\lambda}\right)^{1/a-1}$$

$$= \frac{1}{a\lambda}\,\frac{1}{\tau^2}\,\frac{e^{-1/a\lambda\tau}}{\Gamma(\frac{1}{a})}\,\left(\frac{1}{a\lambda\tau}\right)^{1/a-1} = a\lambda\left(\frac{1}{a\lambda\tau}\right)^{1/a+1}\,\frac{e^{-1/a\lambda\tau}}{\Gamma(\frac{1}{a})} \tag{38}$$

$$E(\tau) = \int_0^\infty a\lambda\tau\,\left(\frac{1}{a\lambda\tau}\right)^{1/a+1}\,\frac{e^{-1/a\lambda\tau}}{\Gamma(\frac{1}{a})}\,d\tau = \frac{1}{\lambda(1-a)} \tag{39}$$

Both the mean rate and the mean interval length agree with earlier results.

Now consider an ensemble of Poisson processes, with intensities distributed according to $p_\beta(\beta)$ and thus having a mean intensity of λ. The interval lengths are distributed by $p_\tau(\tau)$ and the mean interval length is $\frac{1}{\lambda(1-a)}$.

The mean interval length for a Poisson process with intensity λ will be $1/\lambda$, but, although the mean intensity of the ensemble of such processes is also λ, the mean interval length of the ensemble will be $\frac{1}{\lambda(1-a)}$ which is greater than $\frac{1}{\lambda}$. This results from the spread of $p_\beta(\beta)$ around λ and the fact that the inverse of a mean can be smaller than the mean of the inverse. A simple example of this is

$$\frac{1}{\bar{i}} = \frac{1}{\frac{1}{11}\sum_{i=5}^{15} i} = \frac{1}{10} = .1$$

whereas

$$\overline{\left(\frac{1}{i}\right)} = \frac{1}{11} \sum_{i=5}^{15} \frac{1}{i} = .112$$

therefore

$$\overline{\left(\frac{1}{i}\right)} > \frac{1}{\bar{i}}$$

Thus, the Pólya process can be interpreted as an ensemble of Poisson processes with intensity distributed by $p_\beta(\beta)$ and interval length by $p_\tau(\tau)$. No one sample function will reflect the characteristics of the Pólya process, but such an ensemble defines it with good approximation.

As mentioned in the Introduction the Pólya process was initially studied to possibly explain the sonar echo structure of submarines. It was initially thought that this structure could be described by such a process rather than by a Poisson process due to the pronounced clustering. This clustering effect was assumed to be inherent in the process through the presence of the "contagiousness" factor a, which, upon the occurrence of an event, would influence the time interval to the next event. From the preceding results, it can be shown that this clustering is an ensemble rather than a sample function effect.

The clustering behavior can be seen by comparing the Pólya process, an ensemble of Poisson processes with mean interval length $\frac{1}{\lambda(1 - a)}$, and a Poisson process having the same mean. One would expect to find a greater number of short time intervals in the Pólya process if it exhibits clustering. Thus, for some T,

$$\lambda \int_0^T \frac{d\tau}{(1 + a\lambda\tau)^{1+1/a}} > \lambda(1 - a)\int_0^T e^{-\lambda(1-a)\tau}\, d\tau \tag{40}$$

Integrating gives

$$\frac{1}{a} \ln (1 + a\lambda T) > \lambda(1 - a)\ T \tag{41}$$

Using the identity

$$\ln x > 1 - \frac{1}{x},\ x > 1$$

and the restriction that a < 1 as the mean is involved, it can be shown that Equation (41) is true for

$$T < \frac{1}{\lambda(1 - a)} \tag{42}$$

This means that there will be a greater number of Pólya intervals with value less than the mean than Poisson intervals, if both processes have the same mean interval value $\frac{1}{\lambda(1 - a)}$.

Solving the equation

$$\lambda(1 + a\lambda\tau)^{-(1+1/a)} = \lambda(1 - a)\ e^{-\lambda(1-a)\tau} \tag{43}$$

shows that the distributions intersect at $\tau = \frac{1}{\lambda(3 - a)}$. A sketch of the distribution functions would appear as

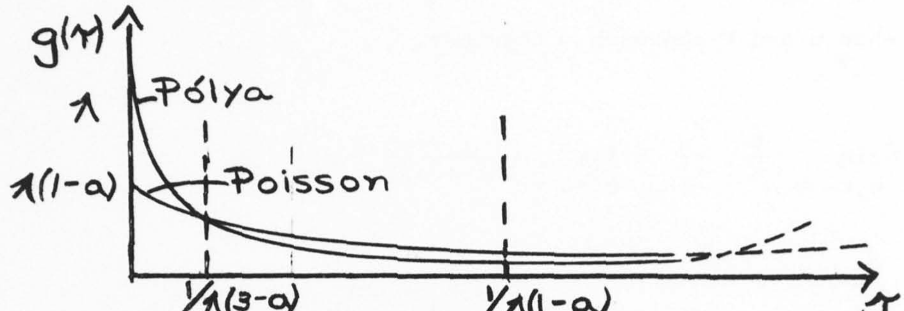


Figure 2 - Probability Distribution Function of τ for Pólya and Poisson Processes Having the Same Mean Interval.

with the Pólya distribution being larger until at least $\tau = \frac{1}{\lambda(1 - a)}$.

In Brian Conolly's experiments, this clustering effect can readily be seen. He tabulated the frequency distribution for 250,000 intervals, in 500 processes, grouped into intervals of one time unit difference. By comparison with a Poisson process having the same mean as the ensemble mean of the Pólya process, it was observed that there were more intervals in a short range for the Pólya process, hence, the clustering effect. An individual sample function would not have given these results as each behaves approximately as a Poisson process with random intensity.

Figures 8-11, using various values of a and λ, show the progression of sample functions of the process in terms of the difference τ, the time interval between the n^{th} and $(n + 1)^{th}$ event. There appears to be a clustering of values which, in view of the above results, represents not the bunching initially expected from the Pólya process, but rather the clustering inherent in Poisson processes with random mean.

An interesting invariant in all of the generated processes is the product of $\frac{\beta}{\lambda}$ and the mean time interval $\frac{t_n}{n}$ as n and t become large. Assuming that n and t approach ∞ together,

$$\lim_{n,t\to\infty} \frac{\beta}{\lambda} \cdot \frac{t_n}{n} = \lim_{n,t\to\infty} \left(\frac{1 + an}{1 + a\lambda t} \right) \cdot \frac{t_n}{n}$$

$$= \lim_{n,t\to\infty} \frac{1}{\lambda} \left(\frac{t_n + ant_n}{\frac{n}{\lambda} + ant_n} \right) = \frac{1}{\lambda} \qquad (44)$$

This has been verified experimentally and results appear in Table 2. The characteristic can be seen by observing Figures 12-14.

It was initially thought that this fixed value per process could be used for parameter estimation, but the transition probability cannot be found for the sample function being considered.

This result, however, does further support the hypothesis that the sample function is a Poisson process with random mean. The product

$$\lim_{n,t\to\infty} \beta \frac{t_n}{n} = 1 \qquad (45)$$

is characteristic of the Poisson process, where $\beta = \frac{n}{t_n}$.

Convergence of the Product of $\frac{\beta}{\lambda}$ and $\frac{t_n}{n}$ to a Constant $\frac{1}{\lambda}$

TABLE 2

a	λ	$\frac{\beta}{\lambda} \times \frac{t_n}{n}$	$\frac{1}{\lambda}$
.5	2.0	.505	.5
.5	2.0	.508	.5
.5	.5	2.01	2.0
.5	.5	1.97	2.0
.4	2.0	.523	.5
.4	2.0	.510	.5
.4	.5	2.00	2.0
.4	.5	2.02	2.0
.3	2.0	.502	.5
.3	2.0	.493	.5
.3	.5	1.96	2.0
.3	.5	2.01	2.0

CORRELATION COEFFICIENT

It was shown previously that the correlation coefficient between two intervals τ_j and τ_{n+1} would be equal to the factor a, if a < .5. 12,000 Pólya processes were generated, with the correlation coefficient between the 1st and 16th intervals being calculated. The results are in Table 3.

Correlation Coefficient for Processes
with Various Values of the Factor a

TABLE 3

a	λ	Correlation Coefficient of $\tau_1 \tau_{16}$
.1	4/30	.0366
.2	4/30	.197
.3	4/30	.398
.4	4/30	.26
.5	4/30	.38
.6	4/30	.408
.7	4/30	.56
.8	4/30	.56
.9	4/30	.686
1.0	4/30	.809

The numerical estimation does not approach the factor a, even below the restriction a <.5, although such a large number of sample points were used.

CONCLUSIONS

Although various methods were employed in an attempt to illuminate the problem of parameter estimation, the non-ergodicity of the process and the discrepancy between theory and experiment in the evaluation of the correlation coefficient make this extremely difficult.

Each process becomes stable as n and t become large and can be interpreted as a Poisson process with random mean. The Pólya process can thus be seen as an ensemble of Poisson processes with clustering occurring only as an ensemble characteristic.

Due to its fluctuating nature, the first segment of each sample function might contain information on clustering tendencies in the sample function. It is being studied.

ACKNOWLEDGMENTS

We would like to thank Ir. Theo Kooij for his suggestion that this process be studied, and for the many stimulating, enlightening discussions that we were able to have with him.

We would also like to thank him for the addition in our Technical Note of his paper, "Integrals Over a Hyperpyramid."

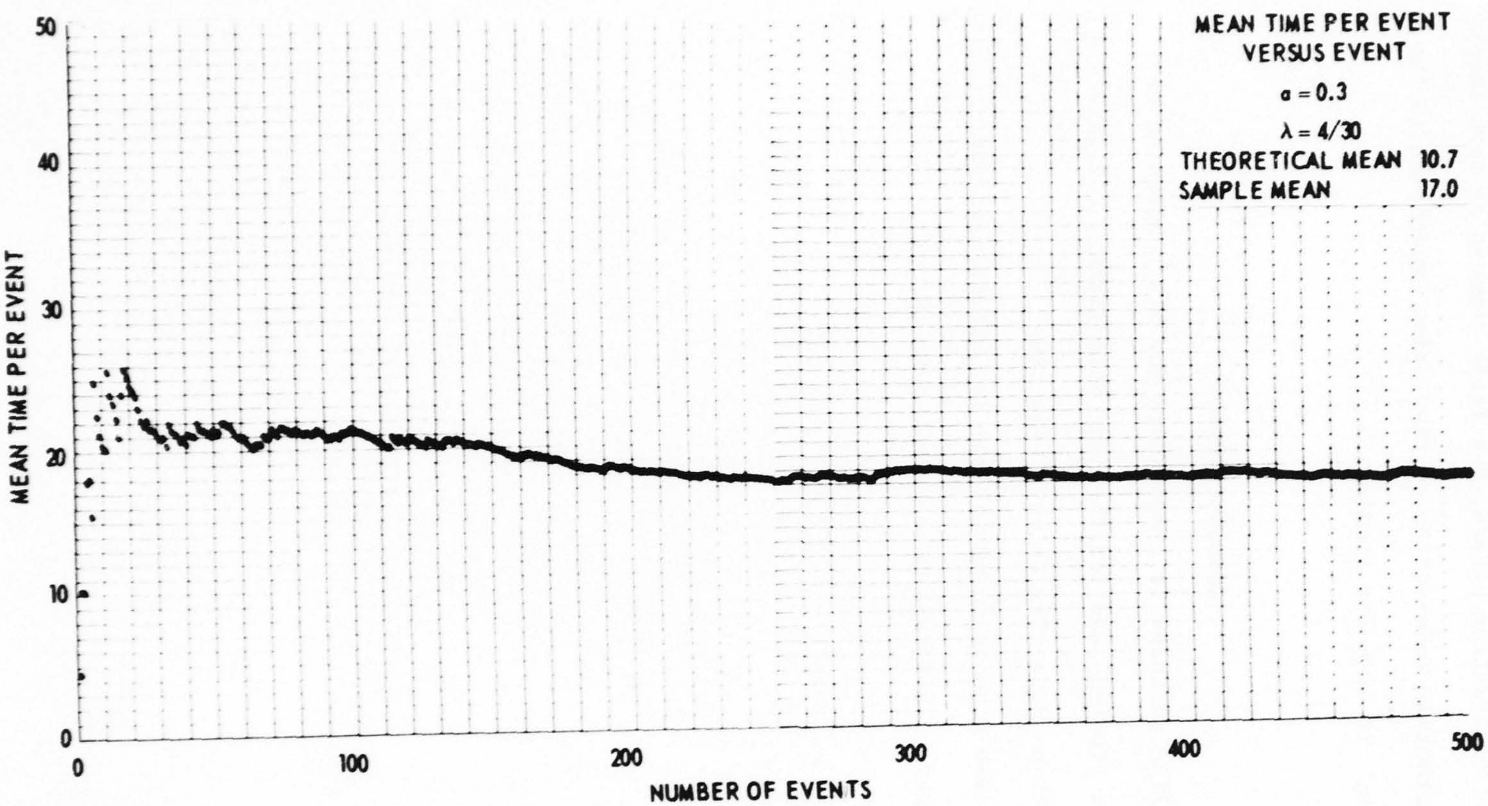

MEAN TIME PER EVENT
VERSUS EVENT
a = 0.3
λ = 4/30
THEORETICAL MEAN 10.7
SAMPLE MEAN 17.0
MEAN TIME PER EVENT
0
10
20
30
40
50
0
100
200
300
400
500
NUMBER OF EVENTS

Figure 3

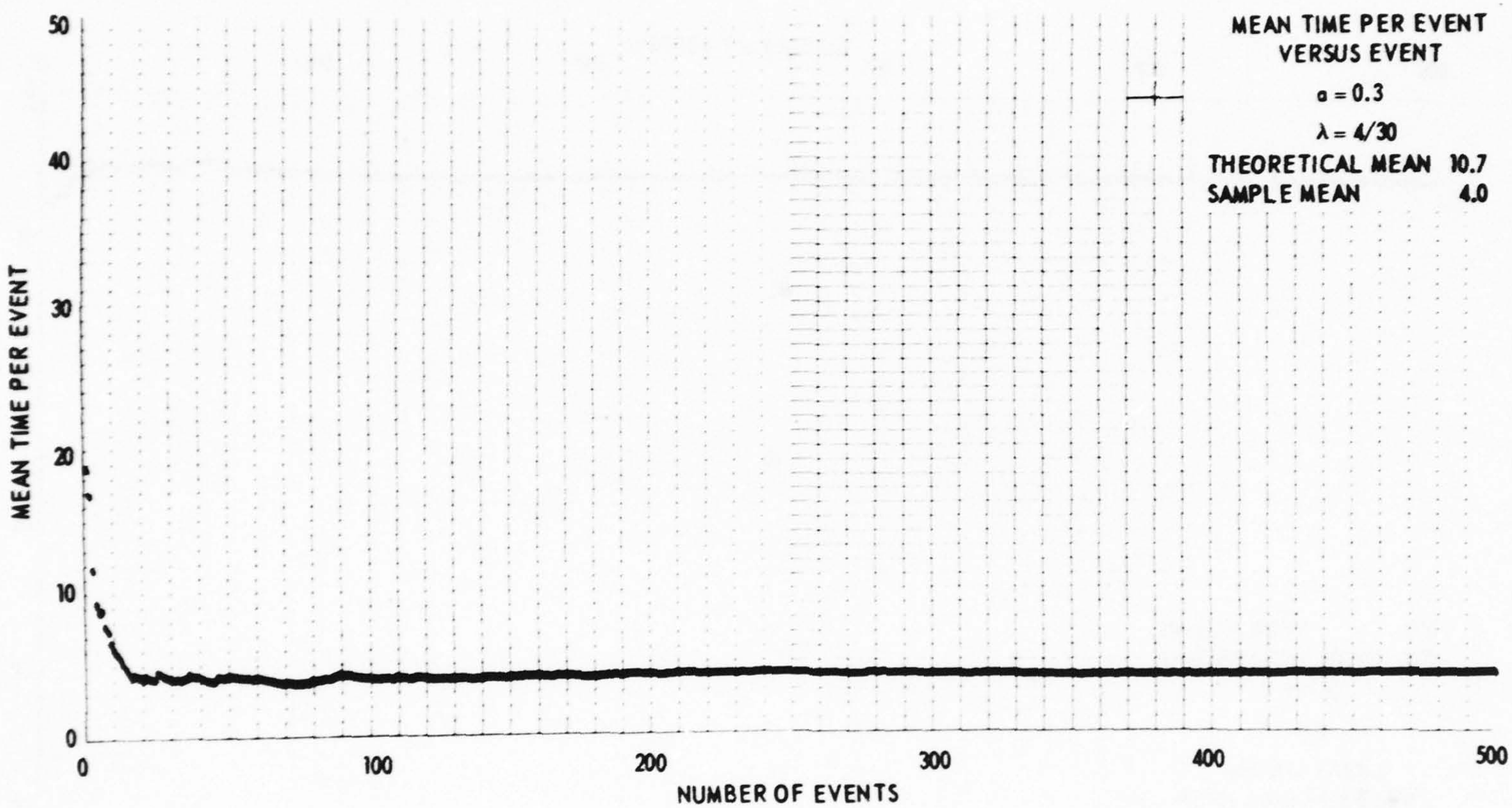

MEAN TIME PER EVENT
VERSUS EVENT
a = 0.3
λ = 4/30
THEORETICAL MEAN 10.7
SAMPLE MEAN 4.0
MEAN TIME PER EVENT
0
10
20
30
40
50
0
100
200
300
400
500
NUMBER OF EVENTS

Figure 4

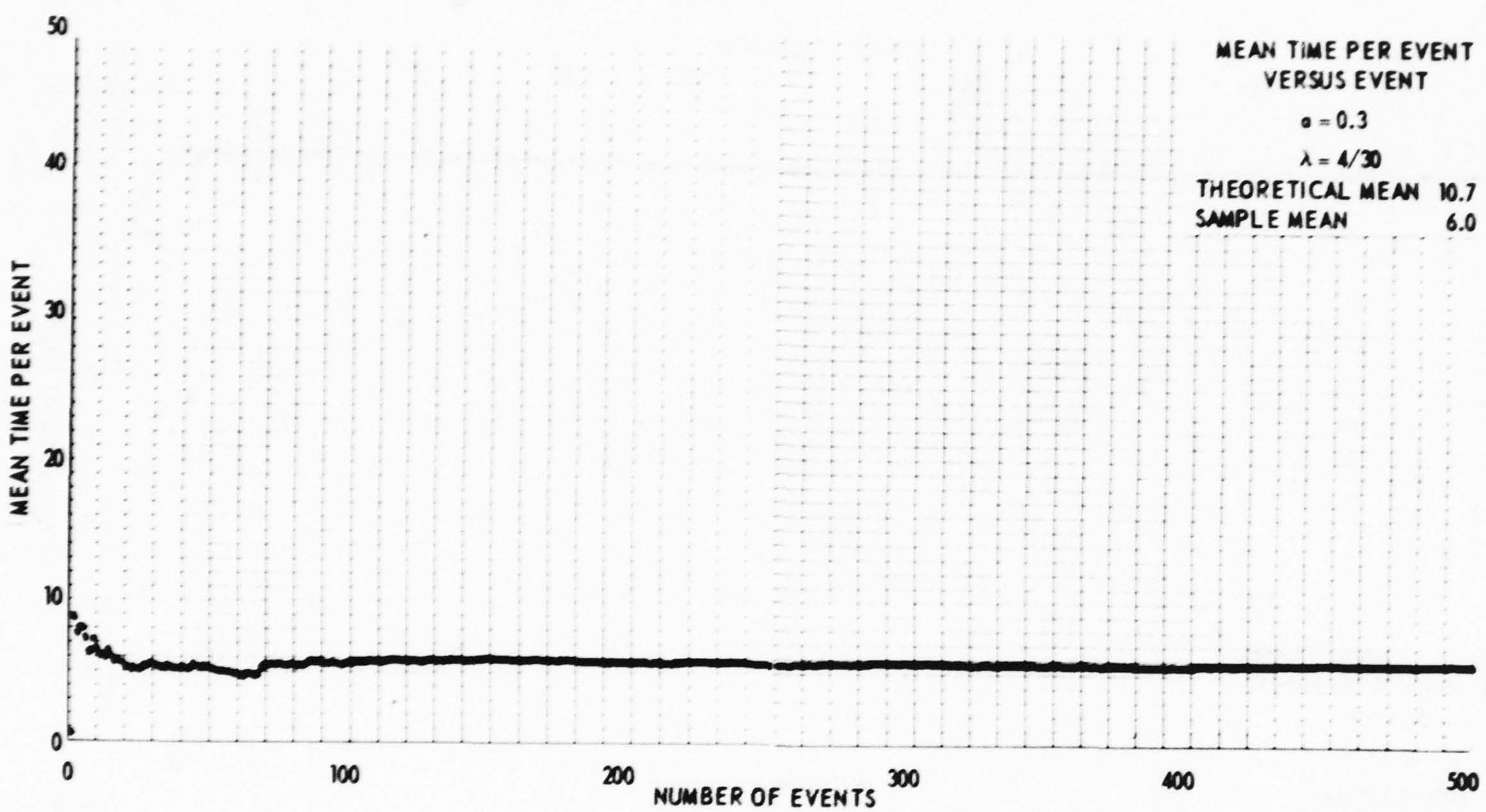
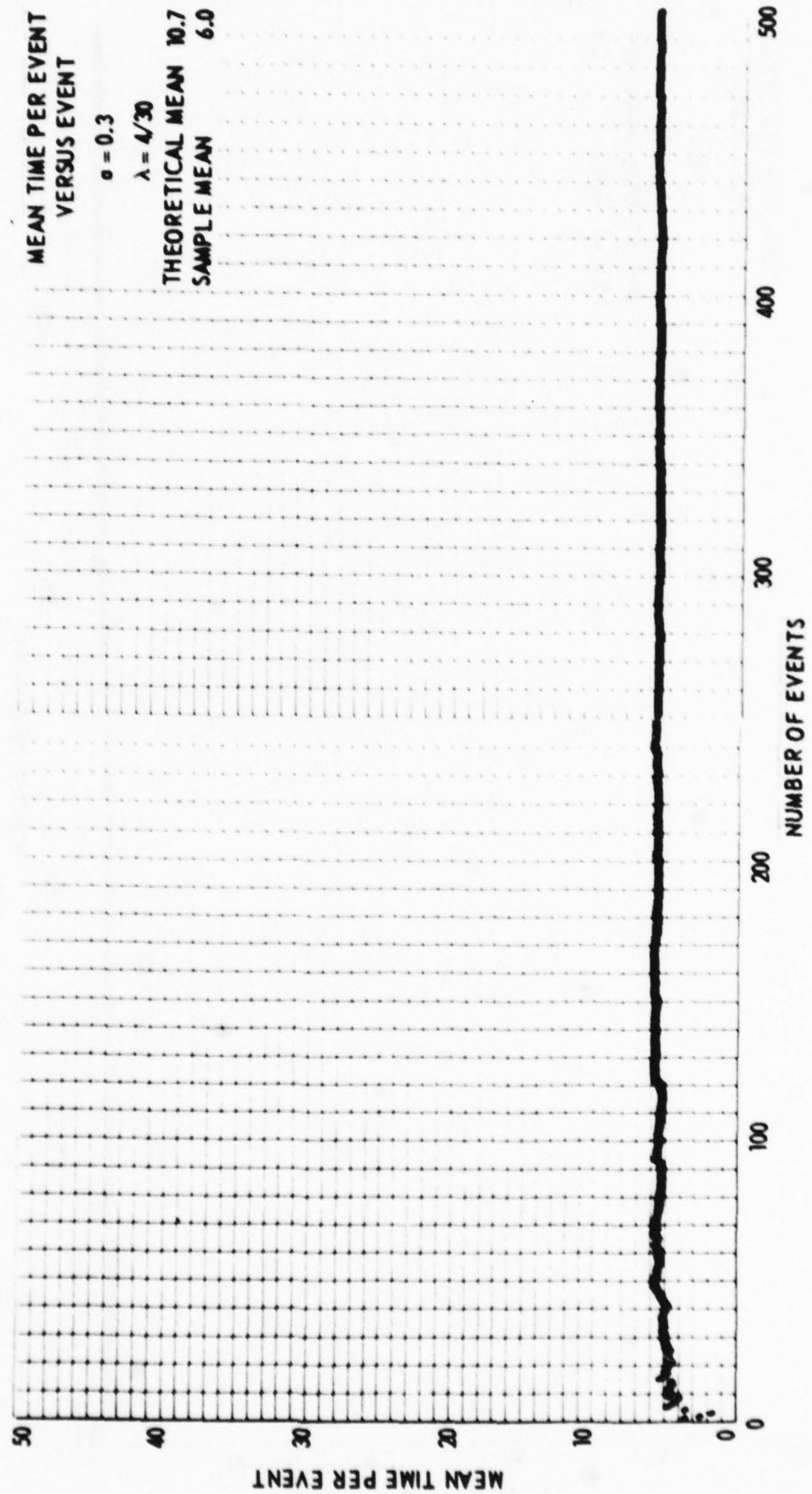

MEAN TIME PER EVENT
VERSUS EVENT
a = 0.3
λ = 4/30
THEORETICAL MEAN 10.7
SAMPLE MEAN 6.0
MEAN TIME PER EVENT
NUMBER OF EVENTS
0
10
20
30
40
50
100
200
300
400
500

Figure 5



MEAN TIME PER EVENT
VERSUS EVENT
a = 0.3
λ = 4/30
THEORETICAL MEAN 10.7
SAMPLE MEAN 6.0
50
40
30
20
10
0
MEAN TIME PER EVENT
0
100
200
300
400
500
NUMBER OF EVENTS

Figure 6

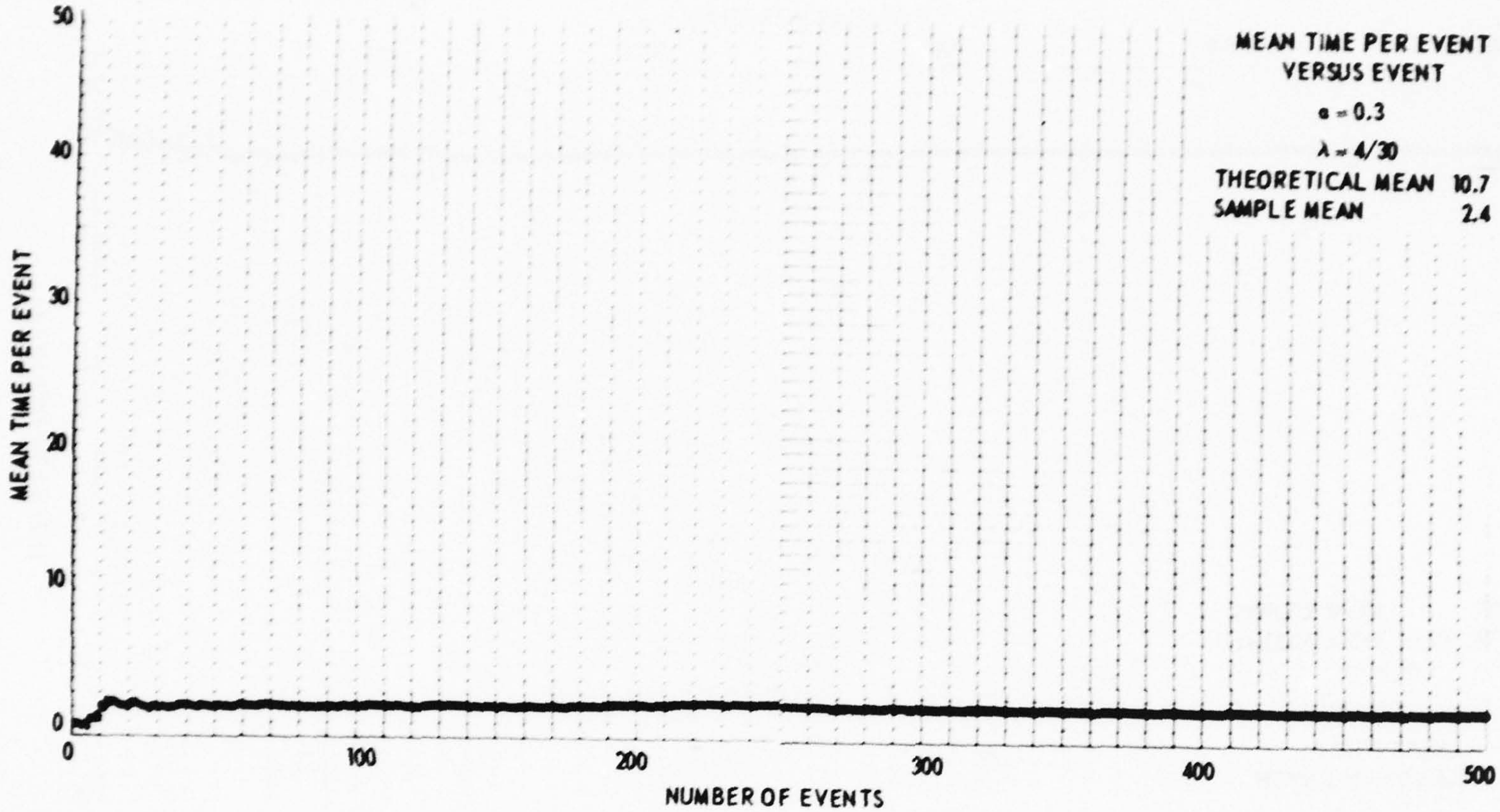

MEAN TIME PER EVENT
VERSUS EVENT
a = 0.3
λ = 4/30
THEORETICAL MEAN 10.7
SAMPLE MEAN 2.4
MEAN TIME PER EVENT
0
10
20
30
40
50
0
100
200
300
400
500
NUMBER OF EVENTS

Figure 7

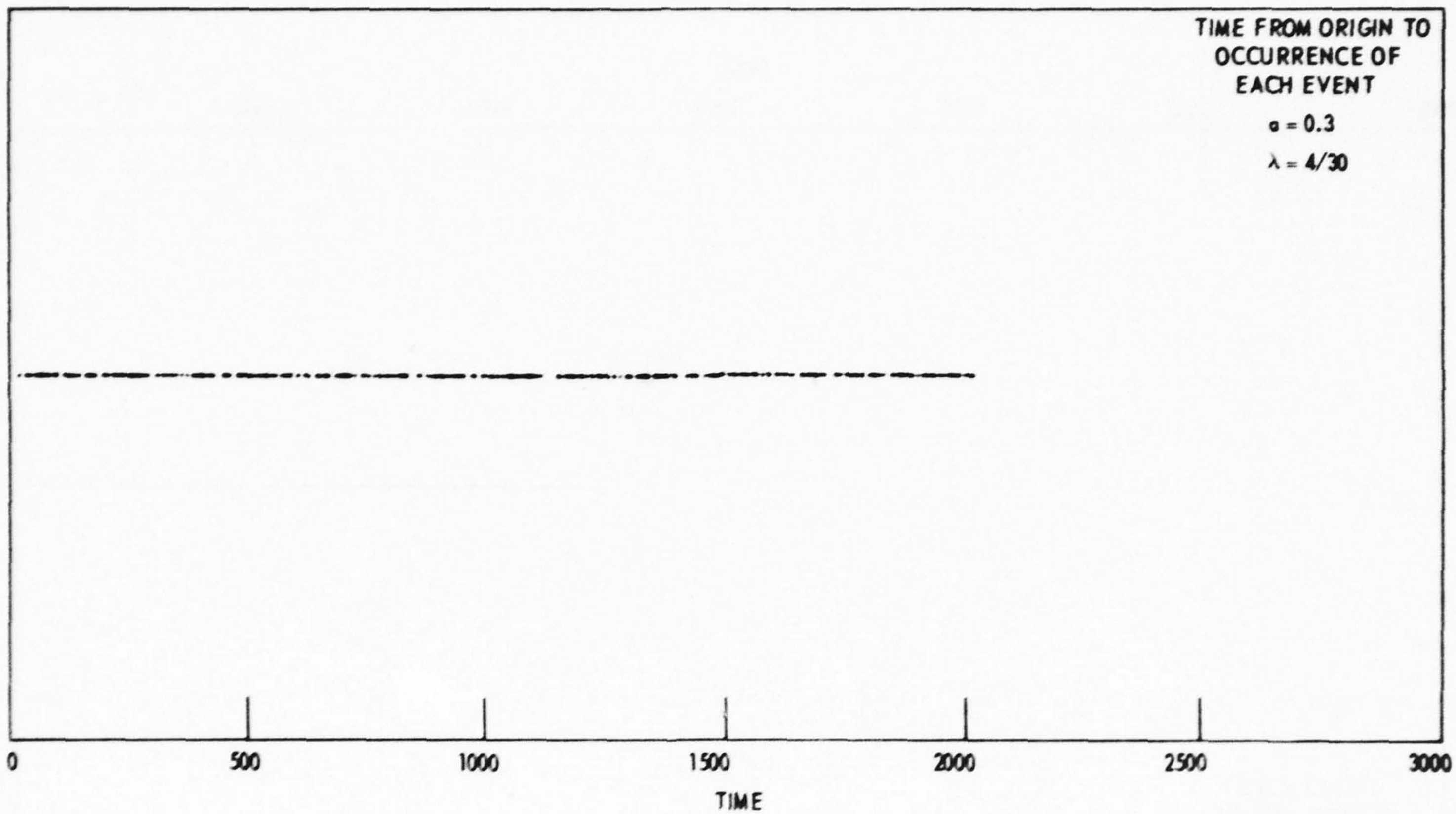

TIME FROM ORIGIN TO OCCURRENCE OF EACH EVENT
a = 0.3
λ = 4/30
0
500
1000
1500
2000
2500
3000
TIME

Figure 8

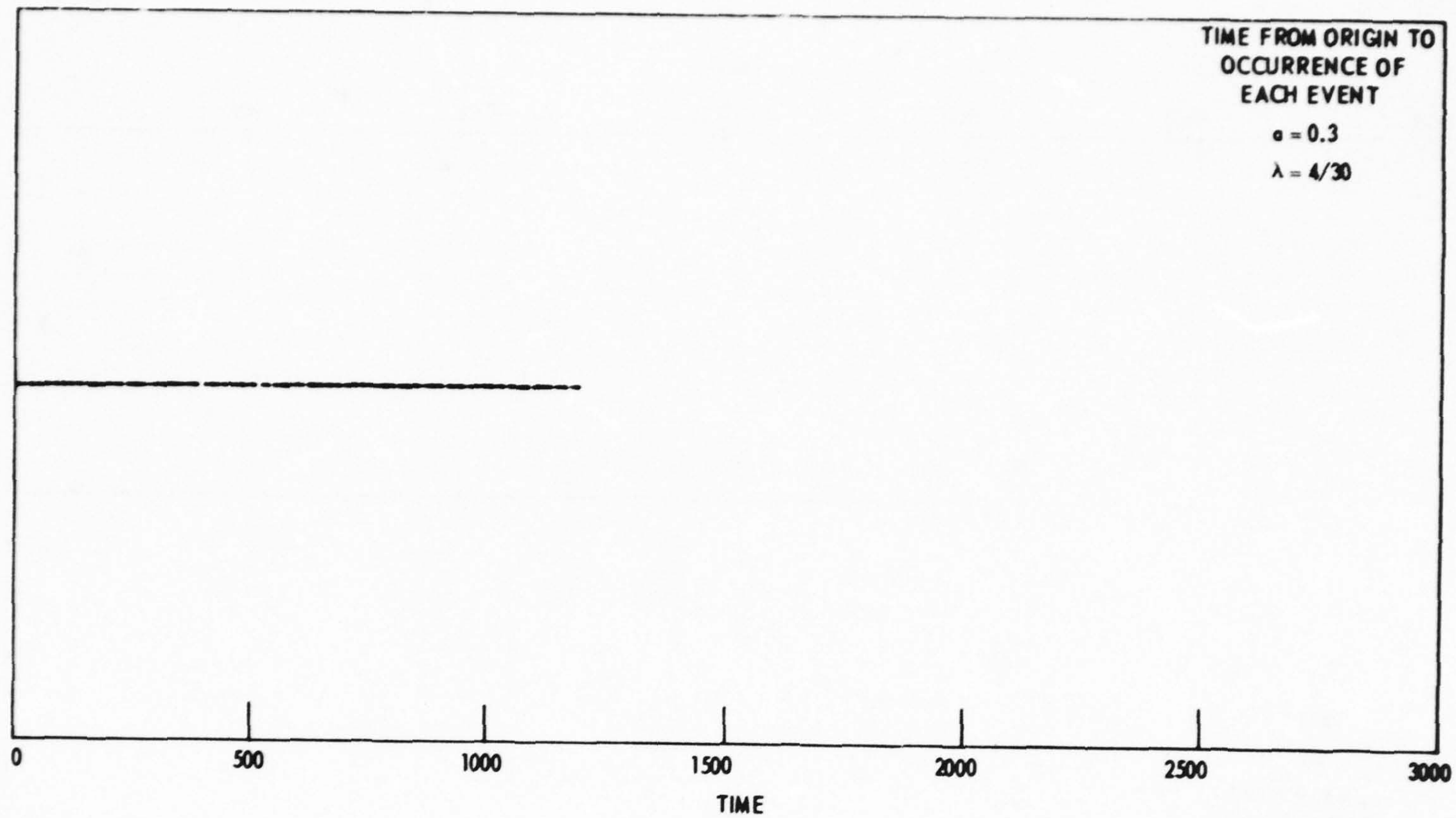

TIME FROM ORIGIN TO
OCCURRENCE OF
EACH EVENT
a = 0.3
λ = 4/30
0
500
1000
1500
2000
2500
3000
TIME

Figure 9

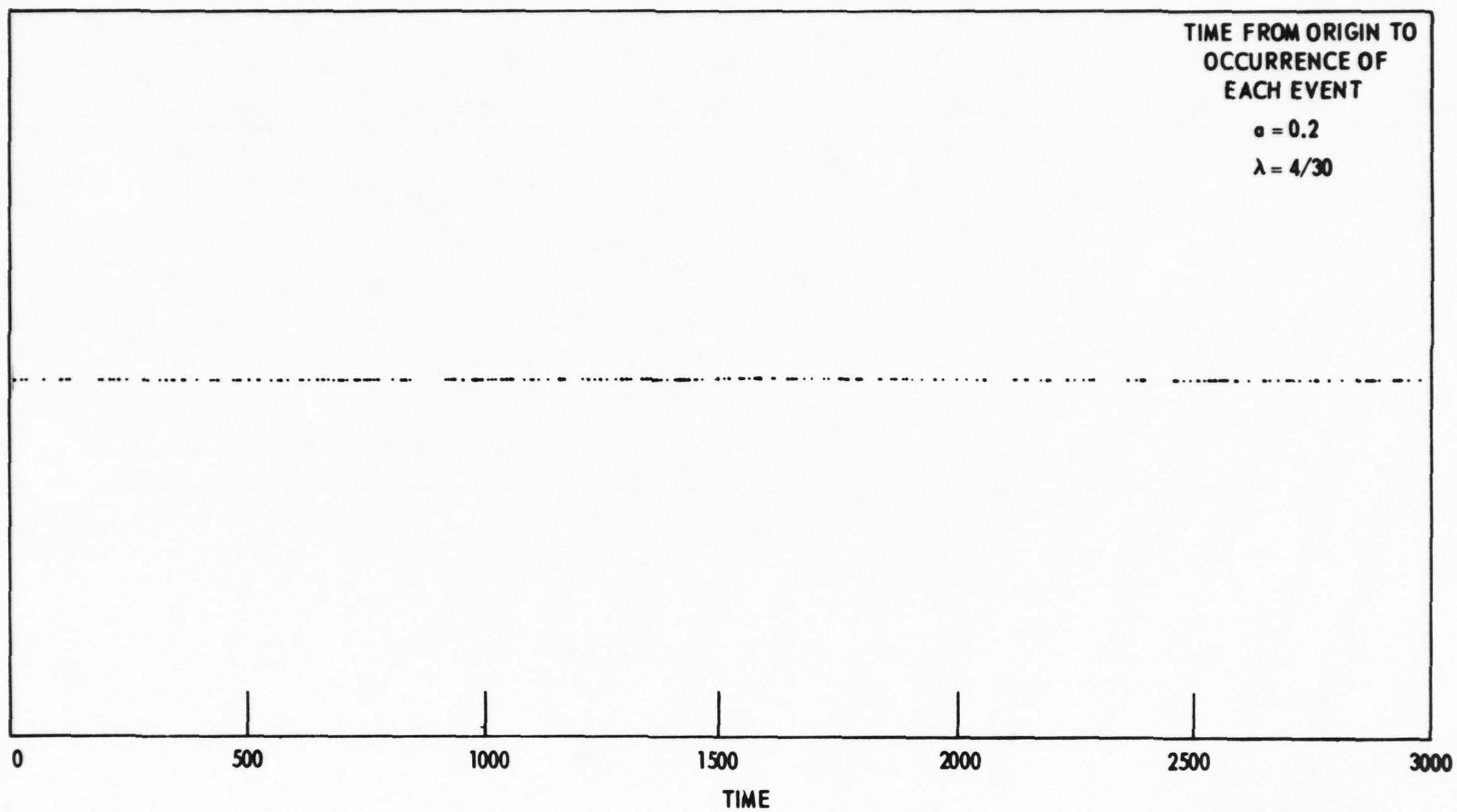

TIME FROM ORIGIN TO
OCCURRENCE OF
EACH EVENT
α = 0.2
λ = 4/30
0
500
1000
1500
2000
2500
3000
TIME

Figure 10

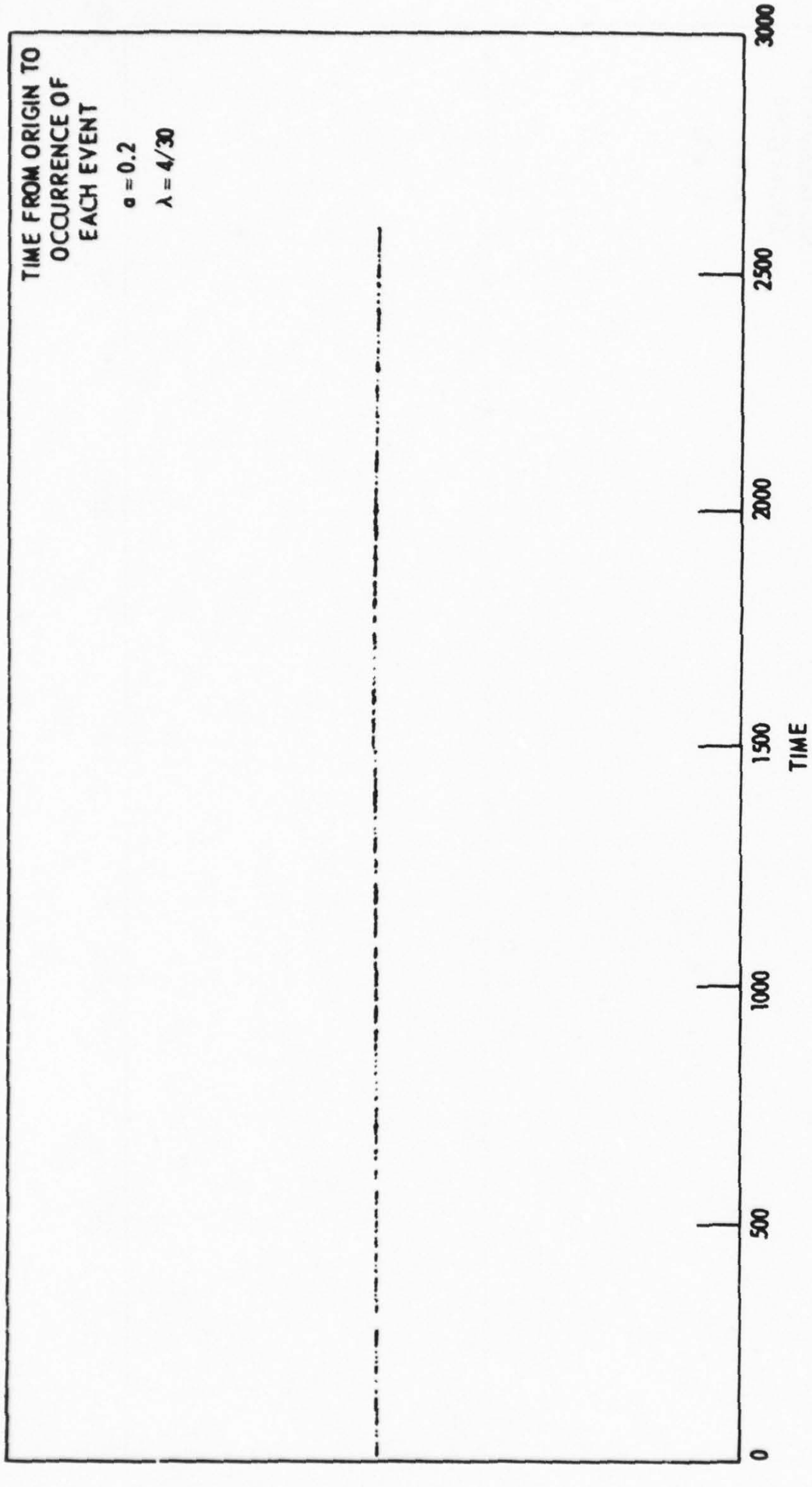
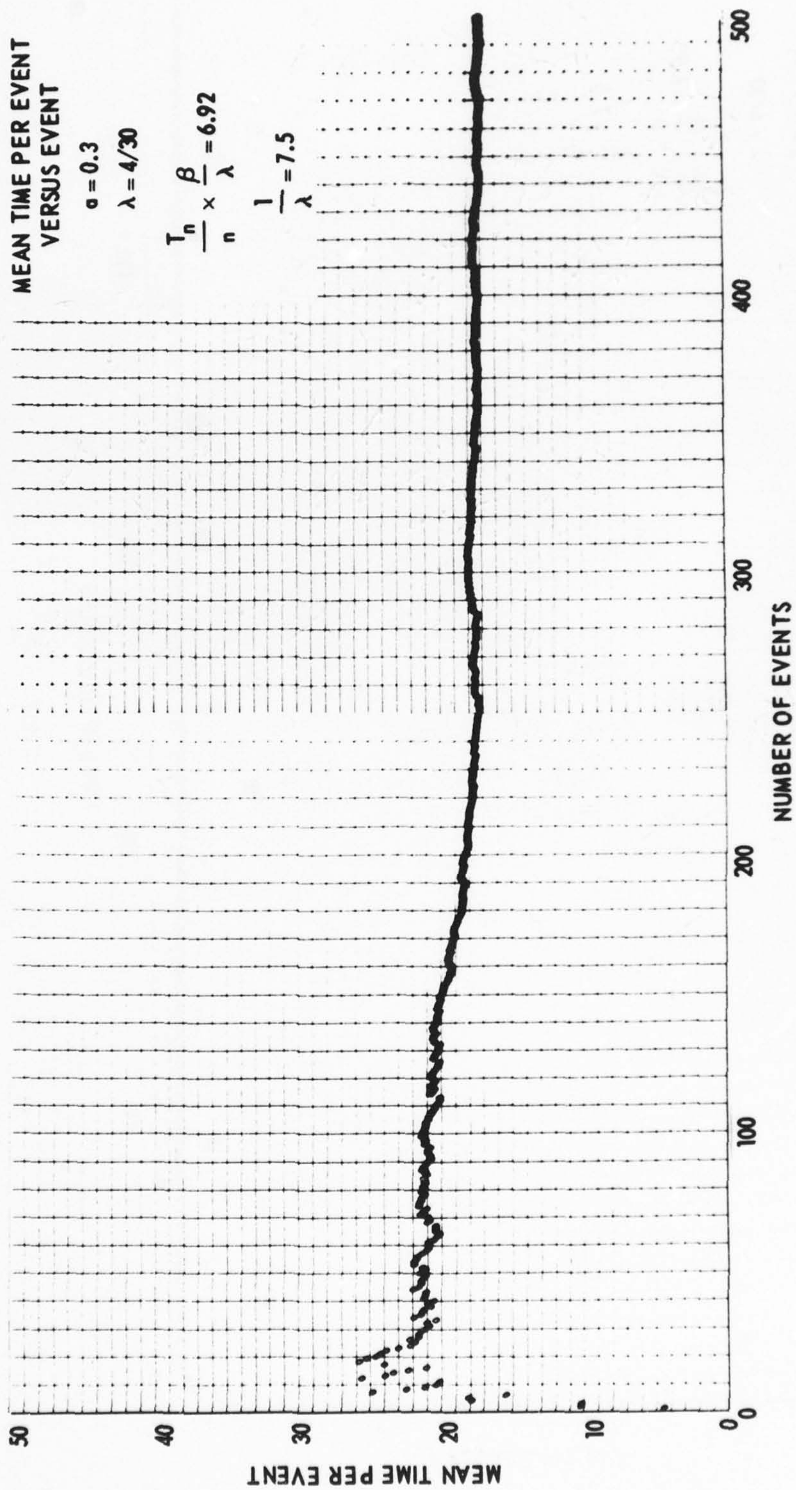

TIME FROM ORIGIN TO
OCCURRENCE OF
EACH EVENT
α = 0.2
λ = 4/30
0
500
1000
1500
2000
2500
3000
TIME

Figure 11



MEAN TIME PER EVENT
VERSUS EVENT
a = 0.3
λ = 4/30
$\frac{T_n}{n} \times \frac{\beta}{\lambda} = 6.92$
$\frac{1}{\lambda} = 7.5$
50
40
30
20
10
0
MEAN TIME PER EVENT
0
100
200
300
400
500
NUMBER OF EVENTS

Figure 12a

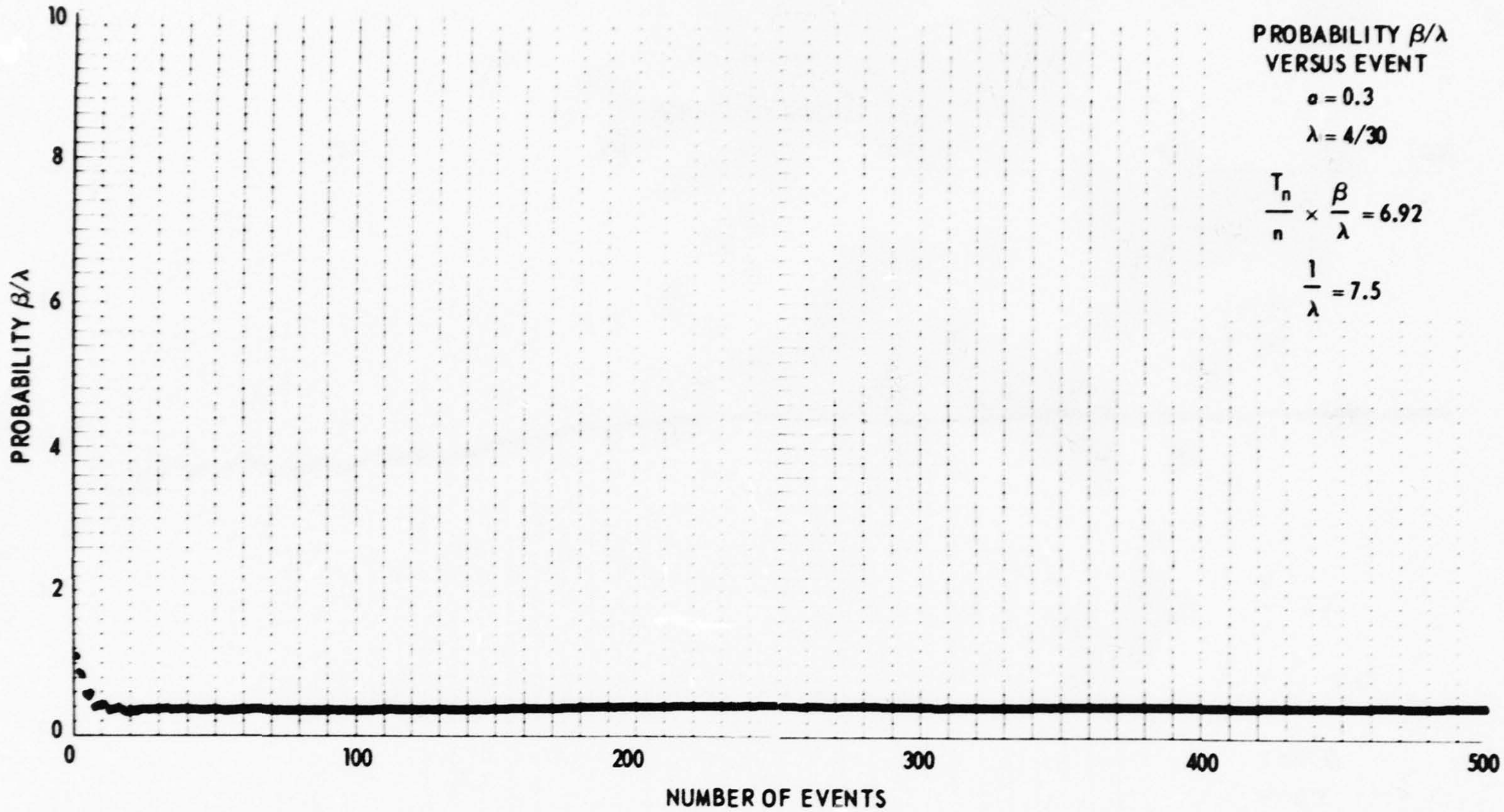

PROBABILITY β/λ
VERSUS EVENT
a = 0.3
λ = 4/30
$\frac{T_n}{n} \times \frac{\beta}{\lambda} = 6.92$
$\frac{1}{\lambda} = 7.5$
PROBABILITY β/λ
0
2
4
6
8
10
0
100
200
300
400
500
NUMBER OF EVENTS

Figure 12b

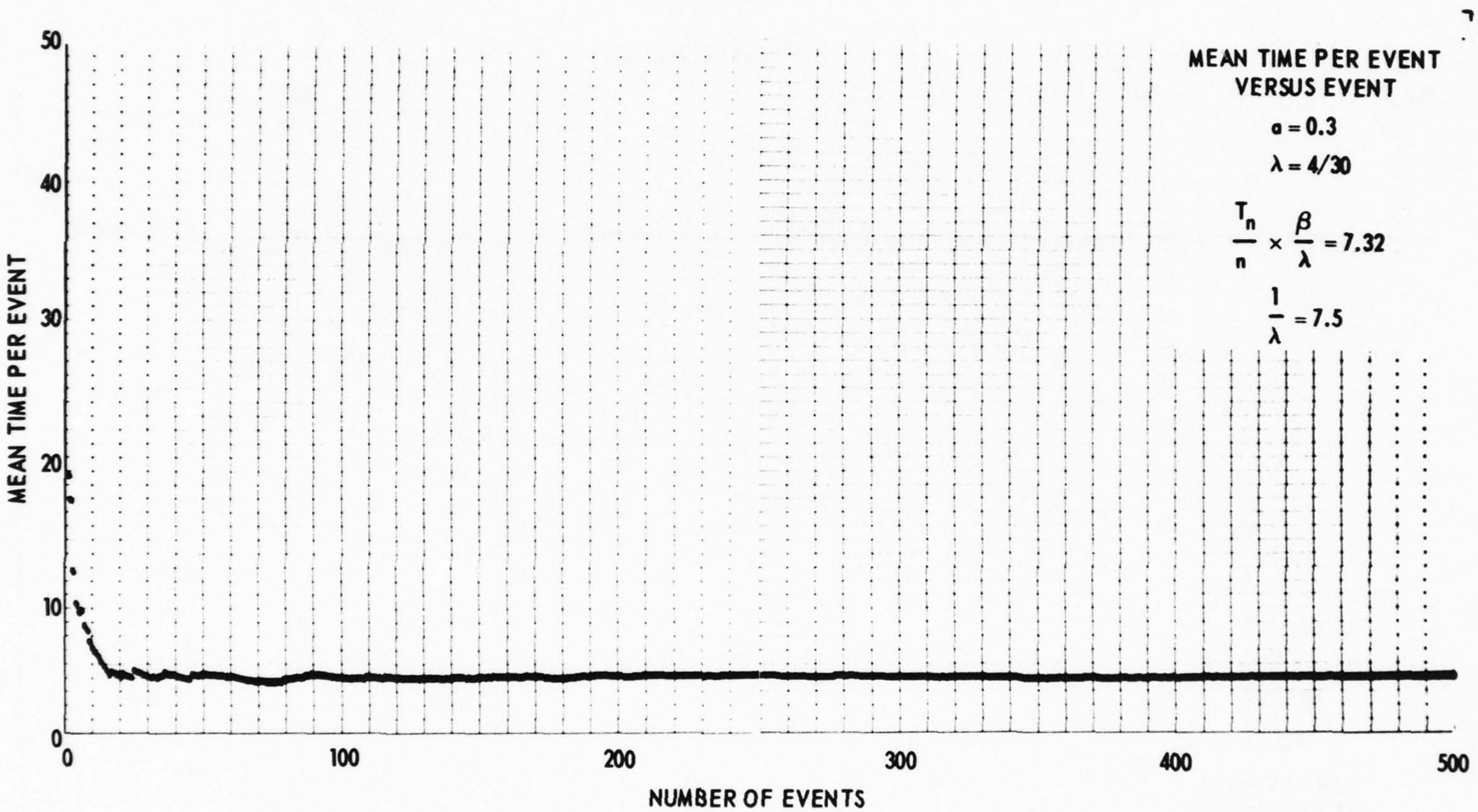

MEAN TIME PER EVENT
VERSUS EVENT
$a = 0.3$
$\lambda = 4/30$
$\frac{T_n}{n} \times \frac{\beta}{\lambda} = 7.32$
$\frac{1}{\lambda} = 7.5$
MEAN TIME PER EVENT
NUMBER OF EVENTS
0
10
20
30
40
50
0
100
200
300
400
500

Figure 13a

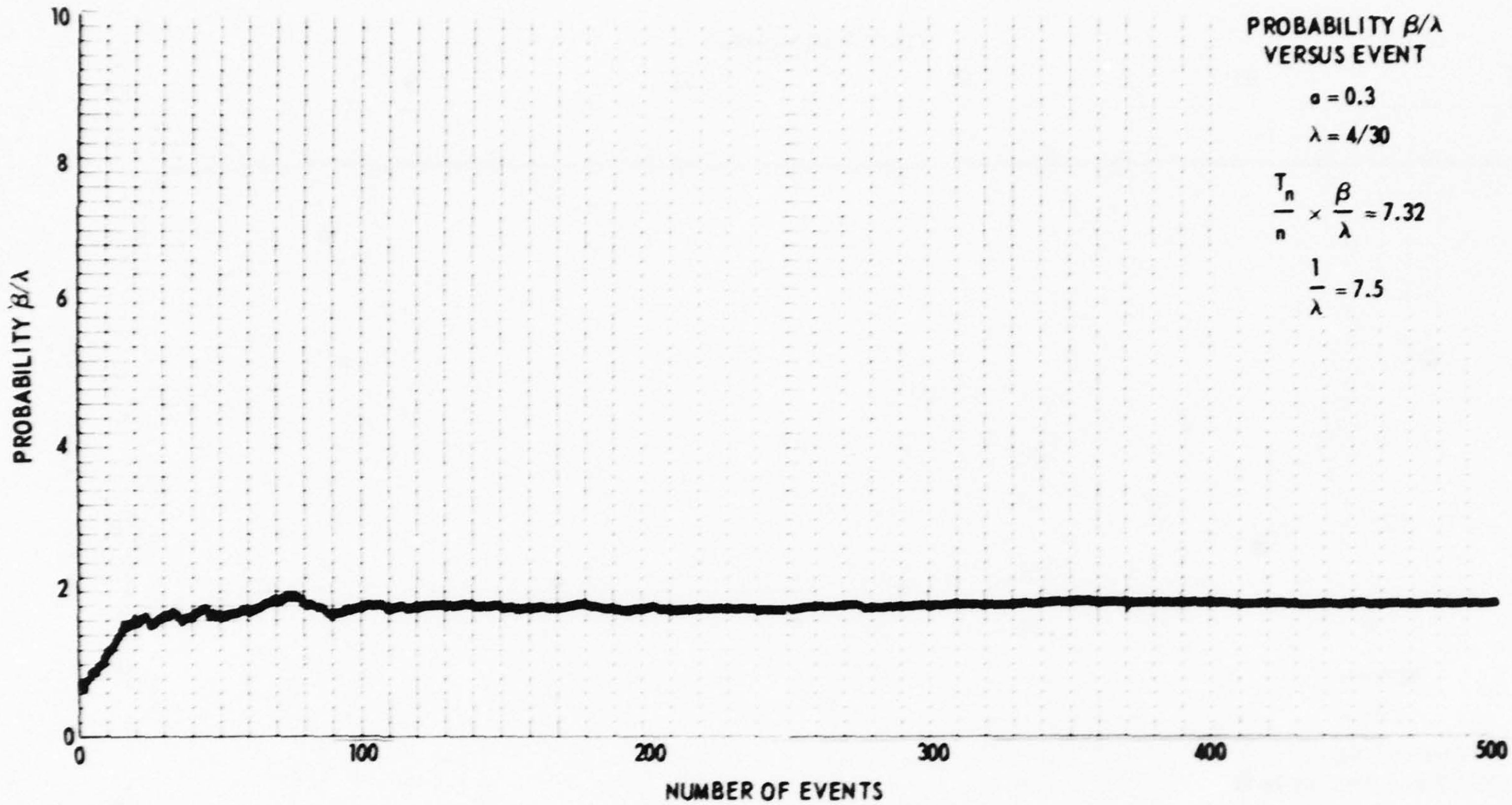

PROBABILITY β/λ
VERSUS EVENT
a = 0.3
λ = 4/30
$\frac{T_n}{n} \times \frac{\beta}{\lambda} = 7.32$
$\frac{1}{\lambda} = 7.5$
PROBABILITY β/λ
NUMBER OF EVENTS
0
2
4
6
8
10
100
200
300
400
500

Figure 13b

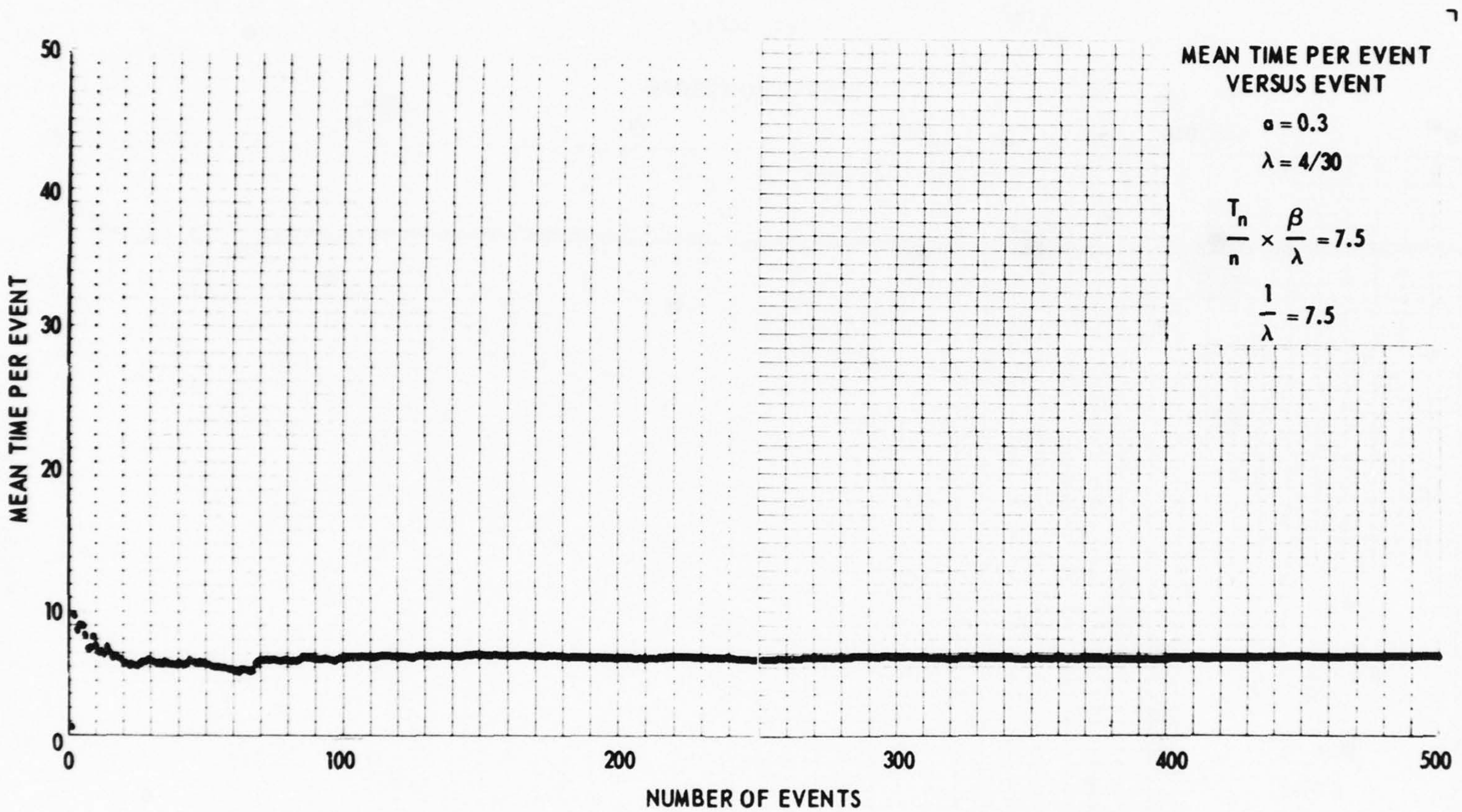

MEAN TIME PER EVENT
VERSUS EVENT
a = 0.3
λ = 4/30
$\frac{T_n}{n} \times \frac{\beta}{\lambda} = 7.5$
$\frac{1}{\lambda} = 7.5$
MEAN TIME PER EVENT
NUMBER OF EVENTS
0
10
20
30
40
50
100
200
300
400
500

Figure 14a

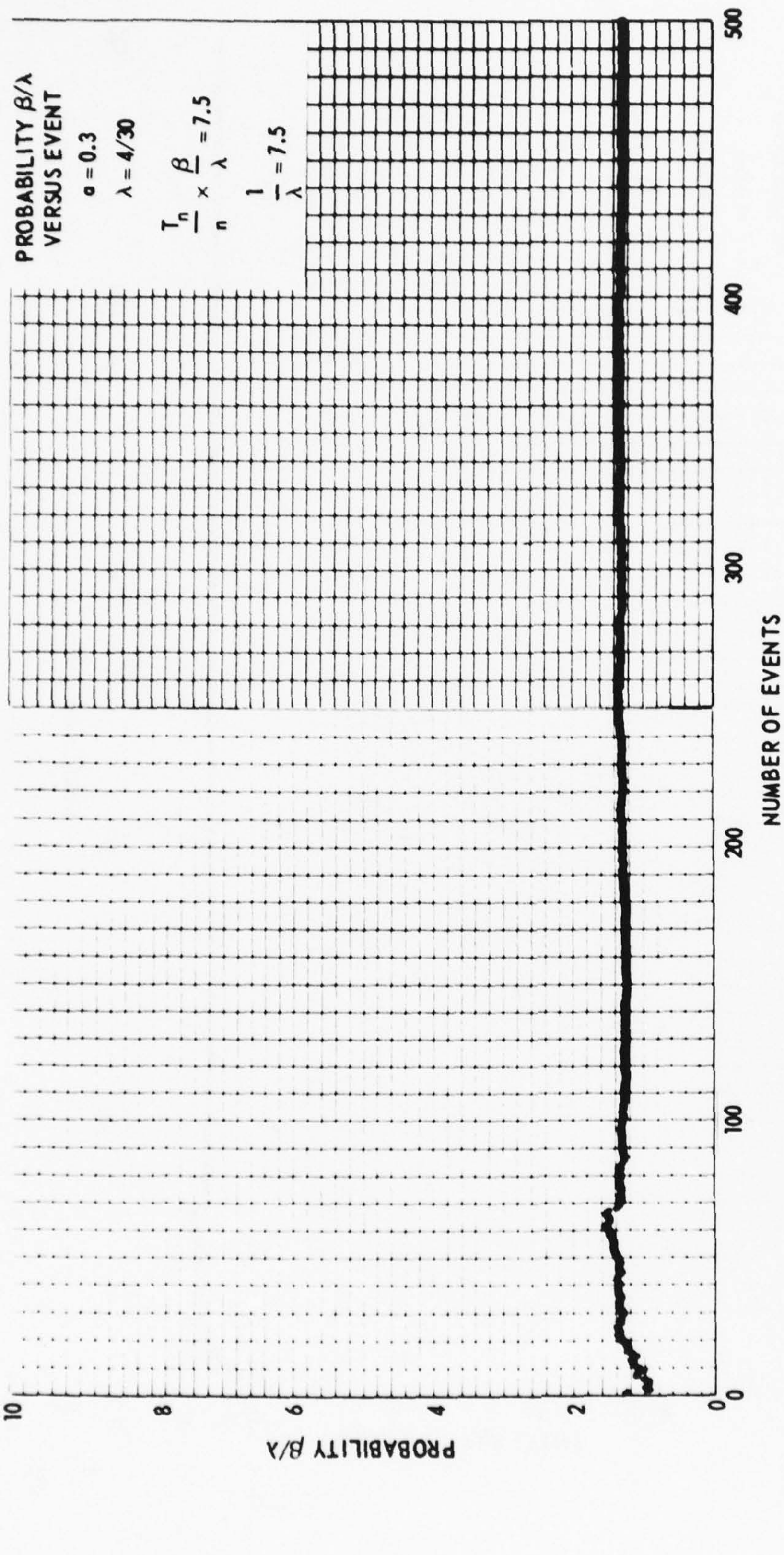

PROBABILITY β/λ
VERSUS EVENT
a = 0.3
λ = 4/30
$\frac{T_n}{n} \times \frac{\beta}{\lambda} = 7.5$
$\frac{1}{\lambda} = 7.5$
PROBABILITY β/λ
0
2
4
6
8
10
0
100
200
300
400
500
NUMBER OF EVENTS

Figure 14b

APPENDIX A

First consider the possible situation of having no events occur in the interval $(0,t)$. If a division is made into subintervals such that $\delta t = \frac{t}{s}$, then the probability that no event occurs in the r^{th} interval is

$$1 - \beta_o(t_r)\ \delta t = 1 - \frac{\lambda \delta t}{1 + a\lambda t_r} = 1 - \frac{\lambda \delta t}{1 + a\lambda r \delta t}$$

$$= 1 - \frac{\lambda \frac{t}{s}}{1 + ra\lambda \frac{t}{s}} = 1 - \frac{\lambda t}{s + ra\lambda t} \tag{A1}$$

If every subinterval is to be empty, this probability would be

$$P_o(t) = \prod_{r=1}^{s} \left(1 - \frac{\lambda t}{s + ra\lambda t}\right) \tag{A2}$$

It was found in Equation (3) that the probability of having 0 events in the time t is

$$P_o(t) = \frac{1}{(1 + a\lambda t)^{1/a}} \tag{A3}$$

therefore

$$\lim_{s \to \infty} \prod_{r=1}^{s} \left(1 - \frac{\lambda t}{s + ra\lambda t}\right) = \frac{1}{(1 + a\lambda t)^{1/a}} \tag{A4}$$

As $a \to 0$, $\lim_{a \to 0} (1 + a\lambda t)^{-1/a} = e^{-\lambda t}$, which is the probability for an empty interval t in the Poisson process.

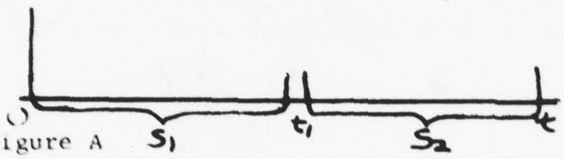


Figure A

In the situation that one event occurs in the interval (0,t) and at time t_1 (Figure A), the probability would be

$$p(t_1)\ \delta t = \prod_{r=1}^{s_1}\left(1 - \frac{\lambda \delta t}{1 + ra\lambda \delta t}\right) \cdot \frac{\lambda \delta t}{1 + a\lambda t_1} \cdot \prod_{r=1}^{s_2} 1 - \frac{1-\lambda(1-a)\delta t}{1+a\lambda(t_1 + r\delta t)} \qquad \text{(A5)}$$

Considering only the last product series:

$$\prod_{r=1}^{s_2}\left(1 - \frac{\lambda(1 + a)\delta t}{1 + a\lambda(t_1 + r\delta t)}\right)$$

$$= \prod_{r=1}^{s_2}\left(1 - \frac{\lambda(1 + a)\dfrac{(t - t_1)}{s_2}}{1 + a\lambda t_1 + a\lambda r\left(\dfrac{t - t_1}{s_2}\right)}\right)$$

$$= \prod_{r=1}^{s_2}\left(1 - \frac{\dfrac{\lambda(1 + a)(t - t_1)}{1 + a\lambda t_1}}{\dfrac{s_2 + ra\lambda(t - t_1)}{1 + a\lambda t_1}}\right)$$

$$= \prod_{r=1}^{s_2}\left(1 - \frac{\dfrac{(1 + a)\ \lambda(t - t_1)}{1 + a\lambda t_1}}{s_2 + r\,\dfrac{a}{1+a}\,\dfrac{(1 + a)\ \lambda(t - t_1)}{1 + a\lambda t_1}}\right) \qquad \text{(A6)}$$

$$= \frac{1}{\left(1 + \frac{a\lambda(t - t_1)}{1 + a\lambda t_1}\right)^{\frac{1+a}{a}}} \tag{A7}$$

Equation (A7) follows from Equation (A6) because of the identity in Equation (A4).

Using this result in the preceding probability expression [Equation (A5)] gives

$$p(t_1)\delta t = \frac{1}{(1 + a\lambda t_1)^{1/a}} \cdot \frac{\lambda}{1 + a\lambda t_1} \cdot \frac{(1 + a\lambda t_1)^{1+1/a}}{(1 + a\lambda t)^{1+1/a}}$$

$$= \frac{\lambda}{(1 + a\lambda t)^{1+1/a}} \tag{A8}$$

which is uniform and independent of t_1. Extending this to the general case where n events occur at times $t_1, t_2 \ldots t_n$ gives

$$p(t_1, t_2 \ldots t_n) = \frac{\lambda^n (1 + a)(1 + 2a) \ldots (1 + \overline{n - 1}a)}{(1 + a\lambda t)^{n+1/a}} \tag{A9}$$

APPENDIX B

INTEGRALS OVER A HYPERPYRAMID

by Theo Kooij

INTRODUCTION

Certain calculations of probability density functions of Poisson and Pólya processes lead to the following type of integrals

$$I_n = \int_0^T d\tau_1 \int_{\tau_1}^T d\tau_2 \int_{\tau_2}^T d\tau_3 \ldots \int_{\tau_{n-1}}^T d\tau_n \, f(\tau_1, \tau_2, \ldots \tau_n)$$

where

$$0 < \tau_1 < \tau_2 < \ldots < \tau_n < T.$$

The compound inequality for the τ_i is the result of the occurrence of sequential events, or of discrete arrival times, in Poisson type processes. Clearly, event number n must occur before event n + 1, and so on.

It appears that a number of manipulations with the boundaries and the order of integration can be done, sometimes leading to forms of I_n that can be integrated more easily or more elegantly. Some of these manipulations and their corresponding considerations will be dealt with here, often in heurestical ways rather than in strict mathematical terms.

THE HYPERPYRAMID

To focus our thoughts, let us first consider the cases I_2 and I_3 for two and three dimensions. Instead of using τ_1, τ_2, τ_3, let us use the more familiar variables x, y, and z. The integral for two dimensions is then

$$I_2 = \int_{x=o}^{T} dx \int_{y=x}^{T} dy\ f(x,y)$$

Since y > x, the area of integration in the X-Y plane is the triangle above the line y = x, bounded by the horizontal line y = T. See Figure 1.

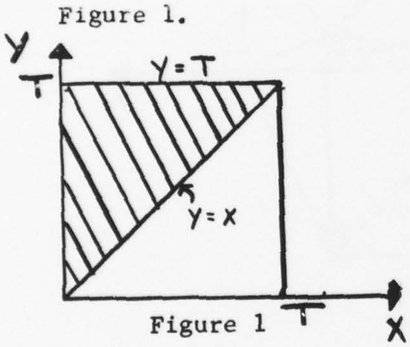


Figure 1

If we let the area become a volume element by adding a third dimension, we obtain

$$I_3 = \int_{x=o}^{T} dx \int_{y=x}^{T} dy \int_{z=y}^{T} dz\ f(x,y,z)$$

The projection of this integration volume on the X-Y plane is the same as the shaded area in Figure 1, since the part containing x and y

is the same. The volume must therefore lie in the vertical prism with a cross section as the shaded area in Figure 1. However, the third integration interval ranges over values of z larger than y, so we have to consider another horizontal prism. Or we might imagine a plane at 45 degrees with the horizontal X-Y plane and with the vertical Y-Z plane, and consider only the volume above this plane. The boundaries thus enclose the pyramid of Figure 2.

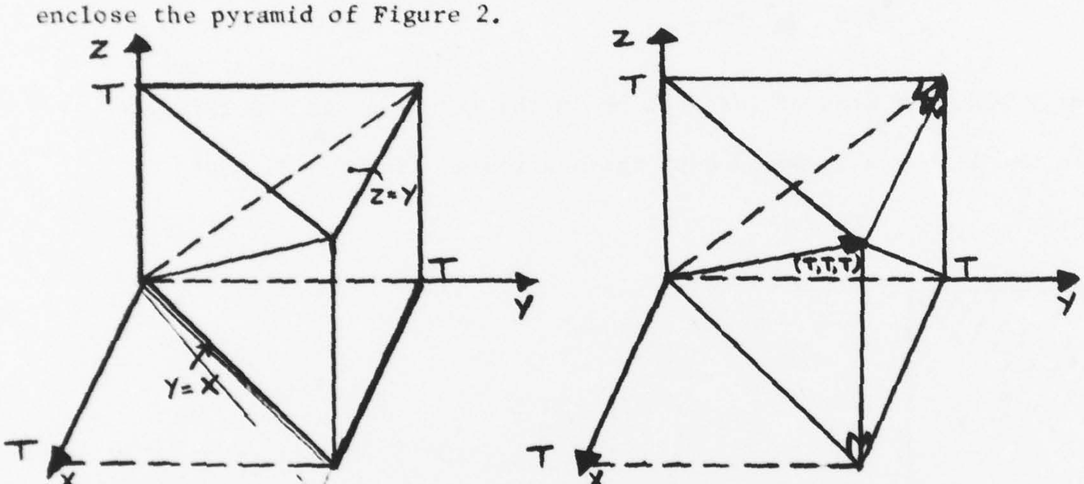


Figure 2 Figure 3

Note that the pyramid contains the main diagonal (T,T,T) as a rib, and lies next to the axis corresponding to the last variable in the sequence $0 < x < y < z$. Cutting the remaining part of the prism once more with a plane through the Y-axis results in two more pyramids, congruent with the first one. See Figure 3. The volume of each pyramid is therefore 1/3 of the prism, or $1/3 \cdot 1/2$ of the total cube volume T^3.

Topologically it can be shown that, by adding another dimension, the three-dimensional top pyramid of Figure 2 can be divided into

four congruent four-dimensional pyramids, all containing the main diagonal (T,T,T,T) as a rib. The volume of each pyramid is thus $\frac{T^4}{4!}$.

In general, the n-dimensional integration volume $0 < \tau_1 < \tau_2 < \ldots < \tau_n < T$ is the hyperpyramid adjoining the τ_n-axis. Its volume is equal to

$$V_n = \int_0^T d\tau_1 \int_{\tau_1}^T d\tau_2 \int_{\tau_2}^T d\tau_3 \ldots \int_{\tau_{n-1}}^T d\tau_n = \frac{T^n}{n!}$$

CHANGES IN THE ORDER OF INTEGRATION

With the picture of the pyramid in mind, let us consider a change in the order of integration over this volume. In the two-dimensional case I_2, the integration over the shaded area of Figure 1 is first performed over $y(x < y < T)$, and this result of the integration over the vertical lines is thereafter integrated horizontally over $x(0 < x < T)$. It follows that alternatively we can integrate first horizontally over $x(y < x < T)$, and then integrate the result of this line integral vertically over $y(0 < t < T)$. Therefore,

$$\int_0^T dx \int_x^T dy\ f(x,y) = \int_0^T dy \int_y^T dx\ f(x,y)$$

A similar procedure can be followed in the three-dimensional case

$$I_3 = \int_0^T dx \int_x^T dy \int_y^T dz\ f(x,y,z)$$

The last two integrals, over y and z, constitute the integrand for the integral over x, and are, therefore, equal to the contribution of the function f(x,y,z) integrated over the triangular area that is obtained by intersecting the pyramid of Figure 2 with the plane x = constant. See Figure 4 for a projection of this area on the Y-Z plane.

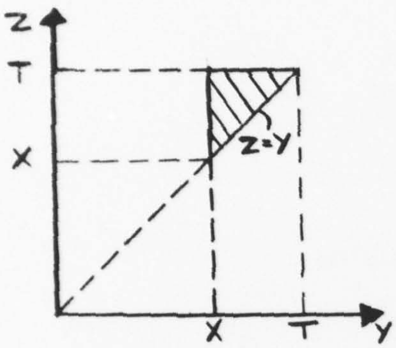


Figure 4

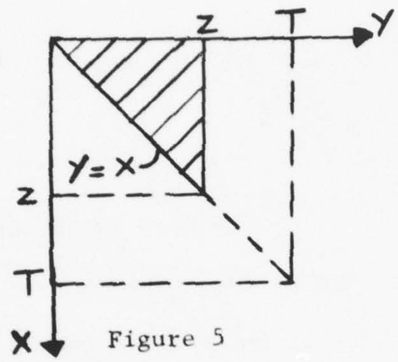


Figure 5

This integral over the triangular area can again be written in two different ways

$$I_{yz} = \int_x^T dy \int_y^T dz\ f(x,y,z) = \int_x^T dz \int_x^z dy\ f(x,t,z)$$

Similarly, we can first integrate over triangles obtained by intersecting the pyramid with planes parallel to the X-Y plane, followed by a final integration over z. An example of the projection of a triangular area on the X-Y plane is given in Figure 5. We find

$$I_{xy} = \int_o^z dx \int_x^z dy\ f(x,y,z) = \int_o^z dy \int_o^y dx\ f(x,y,z)$$

Finally, by integrating first over planes parallel to the X-Z plane, which cut the pyramid into rectangular areas (Figure 6), we find

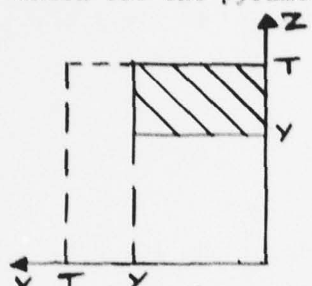


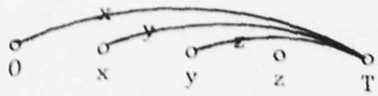
Figure 6

$$I_{xz} = \int_o^y dx \int_y^T dz\ f(x,y,z) = \int_y^T dz \int_o^y dx\ f(x,y,z)$$

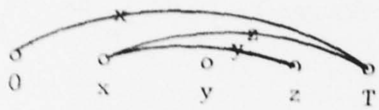
Note that here the two integral boundaries do not change when the order of integration is changed, in accordance with the fact that the boundaries are independent of the integration variables.

We find thus $6 = 3!$ ways of writing I_3 over the three-dimensional pyramid. Generally there are $n!$ ways to express I_n, some of which do not involve changes in the boundaries as was shown above.

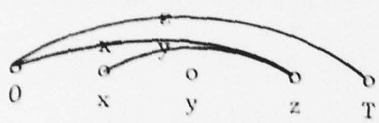
It is instructive to plot the solutions as lines connecting points representing all the integration variables and the two boundaries. See Figure 7 for the three-dimensional case.

(a) 

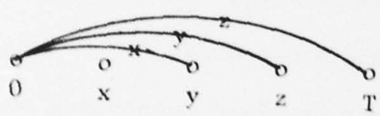
$$\int_0^T dx \int_x^T dy \int_y^T dz \; f(x,y,z)$$

(b) 

$$\int_0^T dx \int_x^T dz \int_x^z dy \; f(x,y,z)$$

(c) 

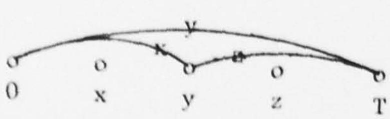
$$\int_0^T dz \int_0^z dx \int_x^z dy \; f(x,y,z)$$

(d) 

$$\int_0^T dz \int_0^z dy \int_0^y dx \; f(x,y,z)$$

(e) 

$$\int_0^T dy \int_0^y dx \int_y^T dz \; f(x,y,z)$$

(f) 

$$\int_0^T dy \int_0^T dz \int_0^y dx \; f(x,y,z)$$

Figure 7

The arches between the points represent the integral invervals. Note the following conditions on the construction of each of the diagrams:

(i) The total number of arches equals the number of integration variables.

(ii) The arches may not cut each other (but two or more may merge at one point).

(iii) Every variable must lie under (not be part of) its corresponding arch. Generalization to the n-dimensional case is obvious.

From the diagrams (e) and (f) in Figure 7 it becomes now directly clear that the corresponding integrals are effectively equal. The two most often used forms of I_n are the types (a) and (d). This gives

$$I_n = \int_0^T d\tau_1 \int_{\tau_1}^T d\tau_2 \int_{\tau_2}^T d\tau_3 \ldots \int_{\tau_{n-1}}^T d\tau_n$$

$$= \int_0^T d\tau_n \int_0^{\tau_n} d\tau_{n-1} \int_0^{\tau_{n-1}} d\tau_{n-2} \ldots \int_0^{\tau_2} d\tau_1$$

The integrals occur whenever variables are integrated out in a multi-variate probability density function of a sequence of arrival times, $p_n(\tau_1, \tau_2, \ldots \tau_n)$, with $0 < \tau_1 < \tau_2 < \ldots < \tau_{n-1} < \tau_n$.

It is then possible to derive probability density functions of lower dimensionality, e.g.,

$$p(\tau_1, \tau_n) = \int_{\tau_1}^{\tau_n} d\tau_2 \int_{\tau_2}^{\tau_n} d\tau_3 \ldots \int_{\tau_{n-2}}^{\tau_n} d\tau_{n-1} \, p_n(\tau_1, \tau_2, \ldots, \tau_n)$$

This integral can subsequently be transformed into different forms by applying the method indicated in this section.

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